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L. I. GOLOVINA
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INDUCTION IN GEOMETRY

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ИЗДАТЕЛЬСТВО МИР

ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Л. И. Головина и И. М. Яглом

ИНДУКЦИЯ В ГЕОМЕТРИИ

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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PREFACE TO THE ENGLISH EDITION

This little book is intended primarily for high school pupils, teachers of mathematics and students in teachers training colleges majoring in physics or mathematics. It deals with various applications of the method of mathematical induction to solving geometric problems and was intended by the authors as a natural continuation of I. S. Sominsky's booklet "The Method of Mathematical Induction" published (in English) by Mir Publishers in 1975. Our book contains 38 worked examples and 45 problems accompanied by brief hints. Various aspects of the method of mathematical induction are treated in them in a most instructive way. Some of the examples and problems may be of independent interest as well.

The book is based on two lectures delivered by Professor I. M. Yaglom to the School Mathematical Circle at the Moscow State University.

The present English edition of the book differs from the Russian original by the inclusion of further examples and problems, as well as additional relevant information on some of the latest achievements in mathematics. It is supplied with a new bibliography in a form convenient for English readers.

L. I. Golovina
I. M. Yaglom

INTRODUCTION: THE METHOD OF MATHEMATICAL INDUCTION

Any reasoning involving passage from particular assertions to general assertions, which derive their validity from the validity of the particular assertions, is called *induction*. The *method of mathematical induction* is a special method of mathematical proof which enables us to draw conclusions about a general law on the basis of particular cases. The principle of this method can be best understood from examples. So let us consider the following example.

EXAMPLE 1. Determine the sum of the first n odd numbers

$$1 + 3 + 5 + \ldots + (2n - 1).$$

Solution. Denoting this sum by $S(n)$, put $n = 1, 2, 3, 4, 5$. We shall then have

$$\begin{aligned} S(1) &= 1, \\ S(2) &= 1 + 3 = 4, \\ S(3) &= 1 + 3 + 5 = 9, \\ S(4) &= 1 + 3 + 5 + 7 = 16, \\ S(5) &= 1 + 3 + 5 + 7 + 9 = 25. \end{aligned}$$

We note that for $n = 1, 2, 3, 4, 5$ the sum of the n successive odd numbers is equal to n^2. We cannot conclude at once from this that it holds for any n. Such a conclusion "by analogy" may sometimes turn out to be erroneous. Let us illustrate our assertion by several examples.

Consider numbers of the form $2^{2^n} + 1$. For $n = 0, 1, 2, 3, 4$ the numbers $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, $2^{2^4} + 1 = 65\,537$ are primes. A notable seventeenth century French mathematician, P. Fermat, conjectured that all numbers of this form are primes. However, in the eighteenth century another great

scientist L. Euler, an academician of Petersburg, discovered that

$$2^{2^5} + 1 = 4\,294\,967\,297 = 641 \times 6\,700\,417$$

is a composite number.

Here is another example. A famous seventeenth century German mathematician, one of the creators of higher mathematics, G. W. Leibnitz, proved that for every positive integer n, the number $n^3 - n$ is divisible by 3, $n^5 - n$ is divisible by 5, $n^7 - n$ is divisible by 7*. On the basis of this, he was on the verge of conjecturing that for every odd k and any natural number n, $n^k - n$ is divisible by k but soon he himself noticed that $2^9 - 2 = 510$ is *not divisible* by 9.

A well-known Soviet mathematician, D. A. Grave, once lapsed into the same kind of error, conjecturing that $2^{p-1} - 1$ is not divisible by p^2 for any prime number p. A direct check confirmed this conjecture for all primes p less than one thousand. Soon, however, it was established that $2^{1092} - 1$ is divisible by 1093^2 (1093 is a prime), i. e. Grave's conjecture turned out to be erroneous.

Let us consider one more convincing example. If we evaluate the expression $991n^2 + 1$ for $n = 1, 2, 3, \ldots$, i. e. for a succession of whole numbers, we shall never get a number which is a perfect square, even if we spend many days or even years on the problem. But we would be mistaken concluding from this that *all* numbers of this kind are not squares, since in fact among the numbers of the form $991n^2 + 1$ there are squares; only the least value of n for which the number $991n^2 + 1$ is a perfect square is very large. Here is the number:

$$n = 12\,055\,735\,790\,331\,359\,447\,442\,538\,767.$$

All these examples must warn the reader against groundless conclusions drawn by analogy.

Let us now return to the problem of calculating the sum of the first n odd numbers. As is clear from what was said above, the formula

$$S(n) = n^2 \qquad (1)$$

cannot be considered as proven, whatever the number of values of n it is checked for, since there is always the possibility that somewhere beyond the range of the cases considered the formula ceases to be true. To make sure that the formula (1) is valid

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* See, for example [21].

for all n we have to prove that however far we move along the natural number sequence, we can never pass from the values of n for which the formula (1) is true to values for which it no longer holds.

Hence, *let us assume that for a certain number* n *our formula is true, and let us try to prove that it is then also true for the next number,* $n + 1$.

Thus, we assume that

$$S(n) = 1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$

Let us compute

$$S(n + 1) = 1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1).$$

By the hypothesis, the sum of the first n terms in the right-hand member of the last equality is equal to n^2, hence

$$S(n + 1) = n^2 + (2n + 1) = (n + 1)^2.$$

Thus, by assuming that the formula $S(n) = n^2$ is true for some natural number n, we were able to prove its validity for the next number $n + 1$. But we checked above that the formula is true for $n = 1, 2, 3, 4, 5$. Consequently, it will also be true for the number $n = 6$ which follows 5, and then it holds for $n = 7$, for $n = 8$, and for $n = 9$, and so on. Our formula may now be considered proven for *any* number of terms. This method of proof is called the *method of mathematical induction.*

Thus, *a proof by the method of mathematical induction consists of the following two parts*:

1°. *A check that the assertion is valid for the least value of* n *for which it remains meaningful**;

2°. *A proof that if the assertion is valid for some arbitrary natural number* n, *then it is also valid for* $n + 1$.

The examples considered above convince us of the necessity of the second part of the proof. The first part of the reasoning is clearly also necessary. It must be emphasized that a proof by the method of mathematical induction definitely requires proof of both parts, 1° and 2°.

A proof that if some proposition is valid for some number n, then it is also valid for the number $n + 1$, on its own, is not enough, since it may turn out that this assertion is not true *for*

---

* It goes without saying that this value of n is not necessarily equal to unity. For instance, any assertion concerning general properties of arbitrary n-gons only makes sense for $n \geqslant 3$.

any integral value of n. For example, if we assume that a certain integer n is equal to the next natural number, i. e. that $n = n + 1$, then, by adding 1 to each side of this equality, we obtain $n + 1 = n + 2$, i. e. the number $n + 1$ is also equal to the next number following it. But it does not follow from this fact that the assertion stated is valid for all n – on the contrary: it is not true for any whole number.

The method of mathematical induction is not necessarily applied strictly in accordance with the above scheme. For example, we sometimes assume that the conjecture under consideration is valid, say, for *two* successive numbers $n - 1$ and n, and have to prove that in this case it is valid for the number $n + 1$ as well. In this case we must begin our reasoning with a check in order to verify that the conjecture is true for the first two values of n, for instance, for $n = 1$ and $n = 2$ (see Examples 17, 18 and 19). As a second step in our reasoning, we sometimes prove the validity of the conjecture for a certain value of n, assuming its validity for *all* natural numbers k *less* than n (see Examples 7, 8, 9, 16).

Let us consider some more examples of applying the method of mathematical induction. The formulas obtained in solving them will be used later on.

EXAMPLE 2. Prove that the sum of the first n natural numbers (let us denote it by $S_1(n)$) is equal to $\frac{n(n+1)}{2}$, i. e.

$$S_1(n) = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \tag{2}$$

Solution. 1°. $S_1(1) = 1 = \frac{1(1+1)}{2}$.

2°. Suppose that

$$S_1(n) = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Then

$$S_1(n+1) = 1 + 2 + 3 + \ldots + n + (n+1) =$$
$$= \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} = \frac{(n+1)[(n+1)+1]}{2},$$

which completely proves the assertion.

EXAMPLE 3. Prove that the sum $S_2(n)$ of the squares of the first n natural numbers is equal to $\frac{n(n+1)(2n+1)}{6}$:

$$S_2(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3}$$

Solution. 1°. $S_2(1) = 1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$.

2°. Assume that

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Then

$$S_2(n+1) = 1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 = \\ = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

and finally

$$S_2(n+1) = \frac{(n+1)[(n+1)+1)][2(n+1)+1]}{6}.$$

PROBLEM 1. Prove that the sum $S_3(n)$ of the cubes of the first n natural numbers equals $\frac{n^2(n+1)^2}{4}$:

$$S_3(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}. \tag{4}$$

EXAMPLE 4. Prove that

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n-1)n = \frac{(n-1)n(n+1)}{3}. \tag{5}$$

Solution. 1°. $1 \times 2 = \frac{1 \times 2 \times 3}{3}$.

2°. If

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n-1)n = \frac{(n-1)n(n+1)}{3},$$

then

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n-1)n + n(n+1) = \\ = \frac{(n-1)n(n+1)}{3} + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

PROBLEM 2. Deduce formula (5) from formulas (2) and (3).

Hint. First show that

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (n-1)n = \\ = (1^2 + 2^2 + 3^2 + \ldots + n^2) - (1 + 2 + 3 + \ldots + n).$$

The method of mathematical induction, being in its essence connected with the notion of number, is most widely employed in arithmetic, algebra and the theory of numbers. Many interesting examples of this kind can be found in the booklet by I. S. Sominsky mentioned in the Preface to this book. But the notion of the whole number is a fundamental one, not only in the theory of numbers, whose subject is the whole numbers and their properties, but in all fields of mathematics. Therefore, the method of mathematical induction is used in many different branches of mathematics. But applications of this method in geometry are especially beautiful, and they form the subject matter of this book. The material is divided into six sections, each being dedicated to a particular type of geometric problem.

Sec. 1. Calculation by Induction

The most natural use of the method of mathematical induction in geometry, one which is close to its use in the theory of numbers in algebra, is its application to solving computational problems in geometry.

EXAMPLE 5. Calculate the side a_{2^n} of a regular 2^n-gon inscribed in a circle of radius R.

Solution. 1°. For $n = 2$ the 2^n-gon is a square; its side $a_4 = R\sqrt{2}$. Then according to the duplication formula

$$a_{2^{n+1}} = \sqrt{2R^2 - 2R\sqrt{R^2 - \frac{a_{2^n}^2}{4}}}$$

we find that the side of a regular octagon $a_8 = R\sqrt{2-\sqrt{2}}$, the side of a regular 16-gon $a_{16} = R\sqrt{2-\sqrt{2+\sqrt{2}}}$, the side of a regular 32-gon $a_{32} = R\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$. We may therefore

assume that for any $n \geqslant 2$ the side of a regular inscribed 2^n-gon

$$a_{2^n} = R\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}_{n-2 \text{ times}}}. \tag{6}$$

2°. Suppose that the side of a regular inscribed 2^n-gon is expressed by formula (6). Then by the duplication formula

$$a_{2^{n+1}} = \sqrt{2R^2 - 2R\sqrt{R^2 - R^2\frac{2 - \underbrace{\sqrt{2 + \ldots + \sqrt{2}}}_{n-2 \text{ times}}}{4}}} =$$

$$= R\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}_{n-1 \text{ times}}},$$

whence it follows that formula (6) is valid for all n.

It follows from formula (6) that the length of a circle of radius R ($C = 2\pi R$) is equal to the limit of the expression $2^n R\sqrt{2 - \underbrace{\sqrt{2 + \ldots + \sqrt{2}}}_{n-2 \text{ times}}}$ as n increases unboundedly and hence

$$\pi = \lim_{n \to \infty} 2^{n-1}\sqrt{2 - \underbrace{\sqrt{2 + \ldots + \sqrt{2}}}_{n-2 \text{ times}}} = \lim_{n \to \infty} 2^n\sqrt{2 - \underbrace{\sqrt{2 + \ldots + \sqrt{2}}}_{n-1 \text{ times}}}.$$

PROBLEM 3. Using formula (6), prove that π equals the limit to which the expression

$$\frac{2}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}\left(1 + \sqrt{\frac{1}{2}}\right)}\sqrt{\frac{1}{2}\left(1 + \sqrt{\frac{1}{2}\left(1 + \sqrt{\frac{1}{2}}\right)}\right)}\cdots}$$

tends as the number of factors (square roots) in the denominator increases unboundedly (Vieta's formula *). The way the factors are formed can be determined from the first three factors (which have been given).

---

* F. Vieta (1540–1603), a well-known French mathematician, one of the creators of algebraic symbols.

Hint. Denote the area of a regular 2^n-gon inscribed in a circle of radius R by S_{2^n}, and its apothem** by h_{2^n}. It then follows from formula (6) that

$$h_{2^n} = \sqrt{R^2 - \frac{a_{2^n}^2}{4}} = \frac{R}{2}\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-1 \text{ times}}$$

and

$$S_{2^n} = \frac{1}{2}(2^n a_{2^n})\, h_{2^n} = 2^{n-2}R^2\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-3 \text{ times}}} = 2^{n-2}Ra_{2^{n-1}}$$

(we assume here that $n \geqslant 3$). Further, we have

$$\frac{S_{2^n}}{S_{2^{n+1}}} = \frac{2^{n-1}a_{2^n}h_{2^n}}{2^{n-1}Ra_{2^n}} = \frac{h_{2^n}}{R} = \cos\frac{180^\circ}{2^n},$$

from which it follows that

$$\frac{S_4}{S_{2^n}} = \frac{S_4}{S_8}\cdot\frac{S_8}{S_{16}}\ldots\frac{S_{2^{n-1}}}{S_{2^n}} = \cos\frac{180^\circ}{4}\cos\frac{180^\circ}{8}\ldots\cos\frac{180^\circ}{2^{n-1}}.$$

Since $S_4 = 2R^2$ and $\lim\limits_{n\to\infty} S_{2^n} = \pi R^2$, $\frac{2}{\pi}$ is equal to the limit of the expression

$$\cos 45^\circ \cos\frac{45^\circ}{2}\cos\frac{45^\circ}{4}\ldots$$

Finally, we make use of the formula

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}.$$

EXAMPLE 6. Find a rule for computing the radii r_n and R_n of the inscribed and circumscribed circles (respectively) of a regular 2^n-gon having a given perimeter P.

Solution. 1°. $r_2 = \frac{P}{8}$, $R_2 = \frac{P\sqrt{2}}{8}$.

2°. Knowing the radii r_n and R_n of the inscribed and circumscribed circles of a regular 2^n-gon of perimeter P, we compute the radii r_{n+1} and R_{n+1} of the inscribed and circumscribed circles of the 2^{n+1}-gon of the same perimeter. Let AB (Fig. 1) be the side of a regular 2^n-gon of perimeter P, O its centre, C the midpoint

** That is the perpendicular distance from the centre to a side (Syn. short radius).

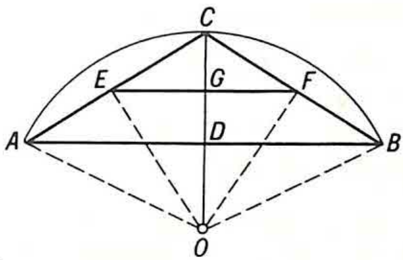

FIG. 1

of the arc AB, and D the midpoint of the chord AB. Let then EF be the midline of the triangle ABC and G its midpoint. Since

$$\angle EOF = \angle EOC + \angle FOC =$$
$$= \frac{1}{2}\angle AOC + \frac{1}{2}\angle BOC = \frac{1}{2}\angle AOB,$$

EF is equal to the side of a regular 2^{n+1}-gon inscribed in a circle of radius OE, the perimeter of this 2^{n+1}-gon being equal to

$$2^{n+1}EF = 2^{n+1}\frac{AB}{2} = 2^n AB,$$

i. e. also equal to P. Hence, $r_{n+1} = OG$ and $R_{n+1} = OE$. Further, it is obvious that $OC - OG = OG - OD$, i. e. $R_n - r_{n+1} = r_{n+1} - r_n$, whence $r_{n+1} = \frac{R_n + r_n}{2}$. Finally, from the right triangle OEC we have $OE^2 = OC \cdot OG$, i. e. $R_{n+1}^2 = R_n r_{n+1}$ and $R_{n+1} = \sqrt{R_n \cdot r_{n+1}}$. Thus, finally we get

$$r_{n+1} = \frac{R_n + r_n}{2} \quad \text{and} \quad R_{n+1} = \sqrt{R_n \cdot r_{n+1}}.$$

Consider the sequence $r_2, R_2, r_3, R_3, \ldots, r_n, R_n, \ldots$ Its terms tend to the radius of a circle of length P, i. e. to $\frac{P}{2\pi}$. In particular, for $P = 2$ we have $r_2 = \frac{1}{4}$ and $R_2 = \frac{\sqrt{2}}{4}$. Putting then $r_1 = 0$ and $R_1 = \frac{1}{2}$, we obtain the following theorem:

If we form the following sequence of numbers

$$0,\ \frac{1}{2},\ \frac{1}{4},\ \frac{\sqrt{2}}{4},\ \frac{\sqrt{2}+1}{8},\ \frac{\sqrt{2\sqrt{2}+4}}{8},\ \frac{\sqrt{2\sqrt{2}+4}+\sqrt{2}+1}{16},\ \ldots,$$

the first two members of which are 0 *and* $\frac{1}{2}$, *and where the remaining terms are alternately equal to the arithmetic mean and to the geometric mean of the two previous terms, then the terms of this sequence tend to* $\frac{1}{\pi}$.

EXAMPLE 7. Determine the sum of the interior angles of an n-gon (which is not necessarily convex).

Solution. 1°. The sum of the interior angles of a triangle is equal to $2d$. The sum of the interior angles of a quadrilateral is equal to $4d$, since any quadrilateral can be split into two triangles (Fig. 2).

2°. Let us assume as already proven that the sum of the interior angles of any k-gon, where $k < n$, is equal to $2d(k-2)$, and consider the n-gon $A_1A_2 \ldots A_n$.

First of all let us prove that in any polygon we can find a diagonal* cutting it into two polygons having a smaller number

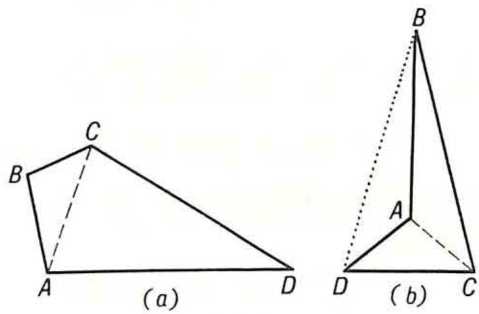

FIG. 2

of sides (for a convex polygon this is clear). Let A, B, C be any three neighbouring vertices of the polygon. Through the vertex B we draw all possible rays filling the interior angle ABC of the polygon to intersect its contour. Two cases are possible:

* Let us here note that a diagonal of a concave polygon can intersect it or lie entirely outside it (see for instance the diagonal BD in Fig. 2, b).

A. The rays all intersect one and the same side of the polygon (Fig. 3, a). In this event the diagonal AC cuts our n-gon into an $(n-1)$-gon and a triangle.

B. The rays don't all intersect one and the same side of the polygon (Fig. 3, b). In this case one of the rays will pass through a certain vertex M of the polygon, and the diagonal BM will break the polygon into two polygons, each with fewer sides.

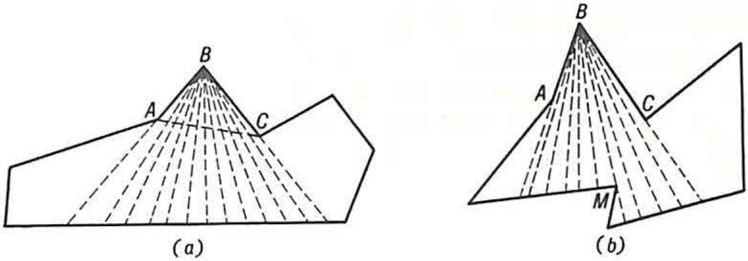

FIG. 3

Let us now return to the proof of our main assertion. In the n-gon $A_1A_2\ldots A_n$ let us draw a diagonal A_1A_k breaking it into the k-gon $A_1A_2\ldots A_k$ and the $(n-k+2)$-gon $A_1A_kA_{k+1}\ldots A_n$. By the assumption made, the sums of the interior angles of the k-gon and the $(n-k+2)$-gon are respectively equal to $2d(k-2)$ and $2d[(n-k+2)-2] = 2d(n-k)$; therefore the sum of the angles of the n-gon $A_1A_2\ldots A_n$ will be equal to

$$2d(k-2) + 2d(n-k) = 2d(n-2),$$

whence follows the validity of our assertion for all n.

As we have just seen in Example 7, in any polygon we can find a diagonal dividing it into two polygons, each with fewer sides. Each of these polygons, provided it is not a triangle, can be broken once again into two polygons with fewer sides, and so on. Hence, each polygon can be divided into triangles by means of non-intersecting diagonals.

EXAMPLE 8. Into how many triangles can an n-gon (convex or concave) be divided by its non-intersecting diagonals?

Solution. 1°. For a triangle this number is equal to 1 (no diagonal can be drawn in a triangle); for a quadrilateral this number is obviously equal to two (see Fig. 2, a and b).

2°. Suppose that we already know, that each k-gon, where $k < n$, is broken by non-intersecting diagonals into $k-2$ triangles

(irrespective of the method of division). Let us consider one of the divisions of the n-gon $A_1A_2 \ldots A_n$ into triangles. Let A_1A_k be one of the diagonals which divides the n-gon $A_1A_2 \ldots A_n$ into the k-gon $A_1A_2 \ldots A_k$ and the $(n-k+2)$-gon $A_1A_kA_{k+1} \ldots A_n$. By the hypothesis, the total number of triangles obtained will be equal to

$$(k-2) + [(n-k+2)-2] = n-2,$$

which proves that our assertion is valid for all n.

PROBLEM 4. Determine the number N of non-intersecting diagonals used in breaking an n-gon into triangles.

Hint. From the fact that the N diagonals and n sides of the n-gon are the sides of $n-2$ triangles (see Example 8) it follows that

$$2N + n = 3(n-2), \quad N = n-3.$$

EXAMPLE 9. Find a rule for calculating the number $P(n)$ of ways in which a convex n-gon can be divided into triangles by non-intersecting diagonals.

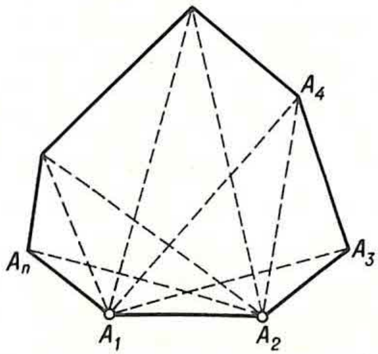

FIG. 4

Solution. 1°. For a triangle this number is obviously equal to unity: $P(3) = 1$.

2°. Let us assume that we have already determined the numbers $P(k)$ for all $k < n$. Let us now find $P(n)$. To do this let us consider the convex n-gon $A_1A_2 \ldots A_n$ (Fig. 4). In any division into triangles, the side A_1A_2 will be a side of one of the triangles of the division. The third vertex of this triangle can coincide with any of the points $A_3, A_4, \ldots, A_n$. The number of ways of

dividing the n-gon in which this vertex coincides with the point A_3 is equal to the number of the ways of dividing the $(n-1)$-gon $A_1A_3A_4 \ldots A_n$ into triangles, i. e. $P(n-1)$. The number of ways in which the vertex coincides with A_4 is equal to the number of ways of dividing the $(n-2)$-gon $A_1A_4A_5 \ldots A_n$, that is $P(n-2) = P(n-2)P(3)$; the number of ways in which it coincides with A_5 equals $P(n-3) \cdot P(4)$, since in this case each division of the $(n-3)$-gon $A_1A_5 \ldots A_n$ can be combined with each division of the quadrilateral $A_2A_3A_4A_5$, and so on. Thus we arrive to the following relation:

$$P(n) = P(n-1) + P(n-2)P(3) + P(n-3)P(4) + \ldots$$
$$\ldots + P(3)P(n-2) + P(n-1). \tag{7}$$

Using this formula successively, we obtain:

$P(4) = P(3) + P(3) = 2,$

$P(5) = P(4) + P(3)P(3) + P(4) = 5,$

$P(6) = P(5) + P(4)P(3) + P(3)P(4) + P(5) = 14$

$P(7) = P(6) + P(5)P(3) + P(4)P(4) + P(3)P(5) + P(6) = 42,$

$P(8) = P(7) + P(6)P(3) + P(5)P(4) + P(4)P(5) + P(3)P(6) +$
$+ P(7) = 132$ and so on.

Note. Using formula (7) we can prove that for any n

$$P(n) = \frac{2(2n-5)!}{(n-1)!(n-3)!}$$

(see, for instance [28]).

PROBLEM 5. Into how many parts is a convex n-gon divided by all its diagonals if no three of them intersect in a single point?

Hint. The convex $(n+1)$-gon $A_1A_2 \ldots A_nA_{n+1}$ is broken by the diagonal A_1A_n into an n-gon $A_1A_2 \ldots A_n$ and a triangle $A_1A_nA_{n+1}$. Considering as known the number $F(n)$ of the parts into which the n-gon $A_1A_2 \ldots A_n$ is divided by its diagonals, let us calculate the number of extra parts obtained by adjoining the vertex A_{n+1} (this number exceeds by unity the number of parts into which the diagonals emanating from the vertex A_{n+1} are divided by the remaining diagonals). In this way we find the relationship

$$F(n+1) = F(n) + (n-1) + 1(n-2) + 2(n-3) + \ldots$$
$$\ldots + (n-3)2 + (n-2)1$$

which, with the aid of formulas (2) and (5) of the Introduction, can be

rewritten in the form

$$F(n+1) = F(n) + (n-1) + \frac{n(n-1)(n-2)}{6} =$$

$$= F(n) + \frac{n^3}{6} - \frac{n^2}{2} + \frac{4n}{3} - 1.$$

Summing the values of $F(n)$, $F(n-1)$, ..., $F(4)$, and using formulas (2), (3) and (4) from the Introduction we get

$$F(n) = \frac{(n-1)(n-2)(n^2-3n+12)}{24}.$$

Sec. 2. Proof by Induction

Several of the propositions of the preceding section can already be considered as examples of the use of the method of mathematical induction for the proof of geometric theorems. For instance, the proposition of Example 7 can be formulated thus: prove that the sum of the angles of an n-gon equals $2d(n-2)$; in Example 8 it was proved that nonintersecting diagonals divide an n-sided polygon into $n-2$ triangles. In this section we will consider further examples of this kind.

EXAMPLE 10. Given: n arbitrary squares. Prove that it is possible to cut them up so that a new square is formed from the resulting parts.

Solution. 1°. For $n = 1$ the assertion requires no proof. Let us prove that for $n = 2$ it is also true. Denote the lengths of the sides of the given squares $ABCD$ and $abcd$ by x and y, respectively; let $x \geqslant y$. On the sides of the square $ABCD$ with side length x (Fig. 5, *a*) mark segments $AM = BN = CP = DQ = \frac{x+y}{2}$ and cut up the square along the straight lines MP and NQ, which, as is easily seen intersect at right angles at the centre O of the square, cutting the square into four equal parts. Now adjoin these pieces to the second square, as shown in Fig. 5, *b*. The figure obtained is also a square, since the angles at the points M', N', P', Q' are supplementary, the angles A', B', C', D' are right angles and $A'B' = B'C' = C'D' = D'A'$.

2°. Suppose that our assertion has already been proved for n squares, and that we are given $n+1$ squares $K_1, K_2, \ldots, K_n, K_{n+1}$.

Take any two of these squares, say, K_n and K_{n+1}. As shown in 1°, by cutting up one of these squares and adjoining the

resulting pieces to the second, we obtain a new square K'. Then, by the assumption, it is possible to cut up the squares K_1, K_2, ..., K_{n-1}, K' so as to form a new square from these parts. QED.

EXAMPLE 11. Given: a triangle ABC, with $n-1$ straight lines CM_1, CM_2, ..., CM_{n-1} drawn through its vertex C, cutting the triangle into n smaller triangles ACM_1, M_1CM_2, ..., $M_{n-1}CB$.

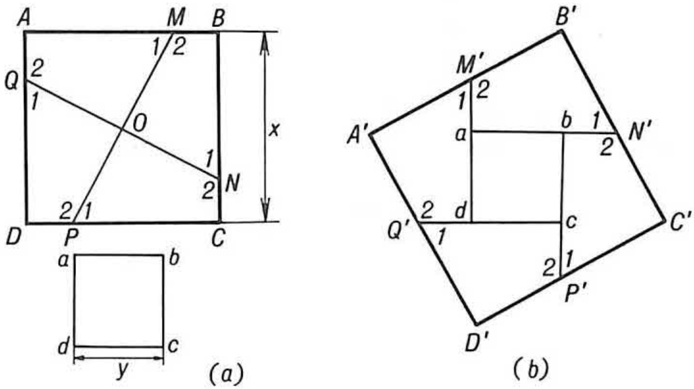

FIG. 5

Denote by r_1, r_2, ..., r_n and ρ_1, ρ_2, ..., ρ_n respectively the radii of the inscribed and circumscribed circles of these triangles (all the circumscribed circles are inscribed within the angle C of the triangle; see Fig. 6, a), and let r and ρ be the radii of the inscribed and circumscribed circles (respectively) of the triangle ABC itself. Prove that

$$\frac{r_1}{\rho_1}\cdot\frac{r_2}{\rho_2}\cdots\frac{r_n}{\rho_n}=\frac{r}{\rho}.$$

Solution. Denote by S the area of the triangle ABC, and by p, its semiperimeter; then, as is well known, $S = pr$. On the other hand, if O is the centre of the circumscribed circle of this triangle (Fig. 6, b), then

$$S = S_{\triangle OAC} + S_{\triangle OCB} - S_{\triangle OAB} = \frac{1}{2}b\rho + \frac{1}{2}a\rho - \frac{1}{2}c\rho =$$
$$= \frac{1}{2}(b + a - c)\rho = (p - c)\rho.$$

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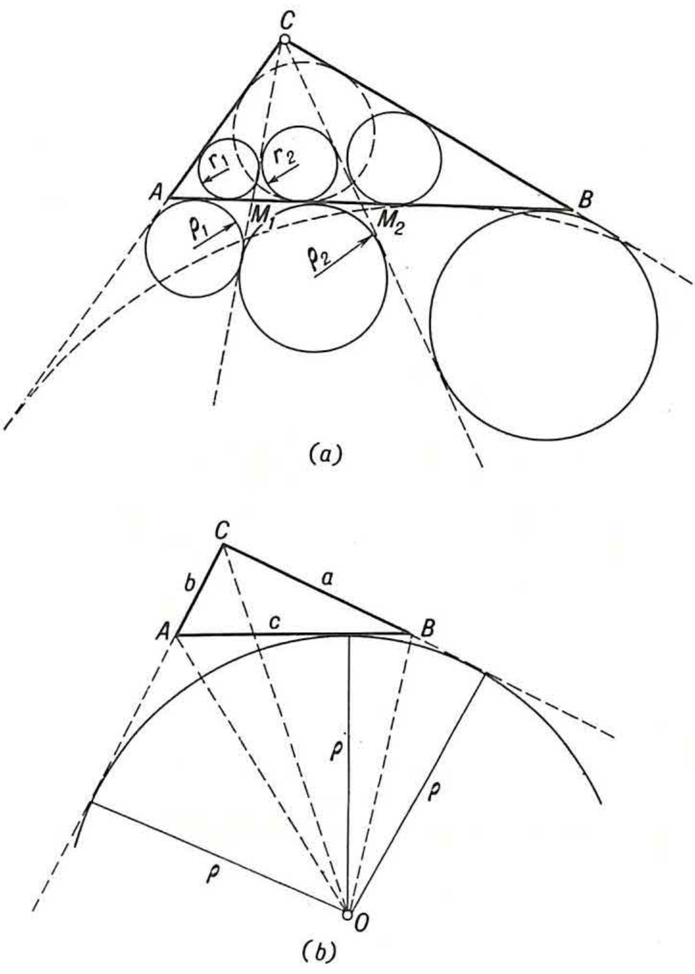

FIG. 6

Hence,

$$pr = (p - c)\,\rho \quad \text{and} \quad \frac{r}{\rho} = \frac{p - c}{p}.$$

Further, from the well-known trigonometrical formulas

$$\tan\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}}$$

and

$$\tan\frac{B}{2} = \sqrt{\frac{(p-a)(p-c)}{p(p-b)}},$$

we have

$$\tan\frac{A}{2}\tan\frac{B}{2} = \sqrt{\frac{(p-b)(p-c)(p-a)(p-c)}{p(p-a)p(p-b)}} = \frac{p-c}{p} = \frac{r}{\rho}. \qquad (8)$$

After these preliminary remarks let us return to the proof of the theorem.

1°. For $n = 1$ the assertion requires no proof. Let us prove that it is true for $n = 2$. In this case the triangle ABC is divided by the straight line CM into two smaller triangles ACM and CMB. By virtue of formula (8)

$$\frac{r_1}{\rho_1}\cdot\frac{r_2}{\rho_2} = \tan\frac{A}{2}\tan\frac{CMA}{2}\tan\frac{CMB}{2}\tan\frac{B}{2} =$$

$$= \tan\frac{A}{2}\tan\frac{CMA}{2}\tan\frac{180° - \angle CMA}{2}\tan\frac{B}{2} = \tan\frac{A}{2}\tan\frac{B}{2} = \frac{r}{\rho}.$$

2°. Suppose that the assertion is proved for $n-1$ straight lines, and that there be given n straight lines $CM_1, CM_2, \ldots, CM_n$, dividing the triangle ABC into $n+1$ smaller triangles ACM_1, $M_1CM_2, \ldots, M_nCB$. Consider two of these triangles, say, ACM_1 and CM_1M_2. As we saw in 1°,

$$\frac{r_1}{\rho_1}\cdot\frac{r_2}{\rho_2} = \frac{r_{12}}{\rho_{12}},$$

where r_{12} and ρ_{12} are the radii of the inscribed and circumscribed circles of the triangle ACM_2. But for n triangles $ACM_2, M_2CM_3, \ldots, \ldots, M_nCB$, we have the equation

$$\frac{r_{12}}{\rho_{12}}\cdot\frac{r_3}{\rho_3}\cdots\frac{r_n}{\rho_n}\cdot\frac{r_{n+1}}{\rho_{n+1}} = \frac{r}{\rho}$$

and, therefore,

$$\frac{r_1}{\rho_1}\cdot\frac{r_2}{\rho_2}\cdots\frac{r_n}{\rho_n}\cdot\frac{r_{n+1}}{\rho_{n+1}} = \frac{r}{\rho}.$$

PROBLEM 6. Let the straight lines CM and CM' divide the triangle ABC in two ways into the two triangles ACM, CMB

and into the triangles ACM', $CM'B$; let r_1, r_2 and r'_1, r'_2 be the radii of their respective inscribed circles. Prove that if $r_1 = r'_1$, then $r_2 = r'_2$ and that a similar property holds for the radii of the circumscribed circles.

Hint. Using the notation of Example 11, prove first of all that

$$\frac{r}{\rho} = 1 - \frac{2r}{h} \quad \text{and} \quad \frac{\rho}{r} = 1 + \frac{2\rho}{h},$$

(where h is the height from vertex C), whence follow the equalities

$$\left(1 - \frac{2r_1}{h}\right)\left(1 - \frac{2r_2}{h}\right) = 1 - \frac{2r}{h} = \left(1 - \frac{2r'_1}{h}\right)\left(1 - \frac{2r'_2}{h}\right)$$

and

$$\left(1 + \frac{2\rho_1}{h}\right)\left(1 + \frac{2\rho_2}{h}\right) = 1 + \frac{2\rho}{h} = \left(1 + \frac{2\rho'_1}{h}\right)\left(1 + \frac{2\rho'_2}{h}\right).$$

PROBLEM 7. Using the notation of Example 11, prove that

$$\frac{r_1 + \rho_1}{R_1} + \frac{r_2 + \rho_2}{R_2} + \ldots + \frac{r_n + \rho_n}{R_n} = \frac{r + \rho}{R},$$

where R_1, R_2, $\ldots$, R_n and R are respectively the radii of the circumscribed circles of the triangles ACM_1, M_1CM_2, $\ldots$, M_nCB and ABC.

Hint. As is well known, $S = pr = (p - c)\,\rho = \dfrac{abc}{4R}$, whence, applying the law of cosines, we obtain

$$\frac{r + \rho}{2R} = \frac{\dfrac{S}{p} + \dfrac{S}{p - c}}{\dfrac{abc}{2S}} = \frac{(a + b)\,[c^2 - (a - b)^2]}{2abc} =$$

$$= \frac{b^2 + c^2 - a^2}{2bc} + \frac{a^2 + c^2 - b^2}{2ac} = \cos CAB + \cos CBA.$$

PROBLEM 8. Given: n circles C_1, C_2, $\ldots$, C_n passing through a point O. We denote by A_1, A_2, $\ldots$, A_n the second points of intersection of circles C_1 and C_2, C_2 and C_3, $\ldots$, C_n and C_1, respectively (Fig. 7, *a*). Let B_1 be an arbitrary point on the circle C_1 which is neither O nor A_1. Draw the secant B_1A_1 to intersect the circle C_2 at point B_2, then the secant B_2A_2 to intersect the circle C_3 at point B_3, and so on (if, for instance, the point B_2

coincides with A_2, then we draw a tangent to the circle C_2 instead of a secant through the point A_2). Prove that the point B_{n+1} which is finally obtained on the circle C_1 coincides with B_1.

Hint. Prove the following preliminary lemma. Let O_1 and O_2 be the centres of the circles C_1 and C_2, which intersect at point O and B_1B_2 a secant through the second point (A_1) of intersection of these circles (see Fig. 7, *b*); then the segments B_1B_2 and O_1O_2 subtend the same angle

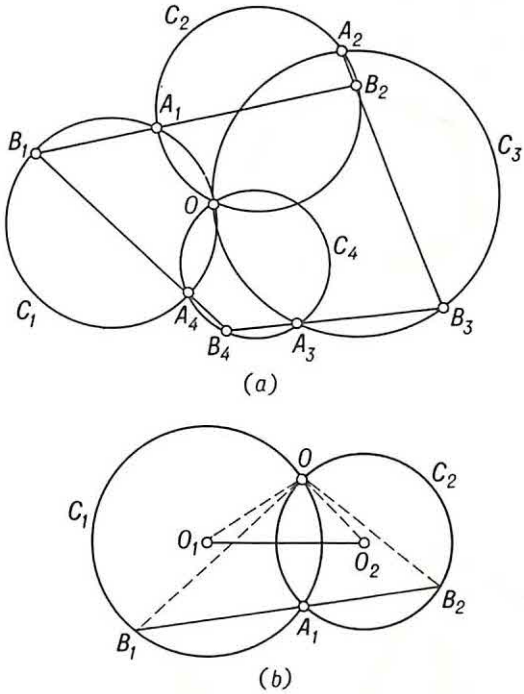

FIG. 7

at the point O. Next prove the proposed theorem for three circles. Then, assuming that the theorem is true for $n-1$ circles, consider n circles $C_1, C_2, \ldots, C_n$. Draw a secant through the point B_{n-1} and the point of intersection of the circles C_{n-1} and C_1; apply the induction assumption to the $n-1$ circles $C_1, C_2, \ldots, C_{n-1}$.

EXAMPLE 12. Prove that any convex n-gon which is not a parallelogram can be enclosed by a triangle whose sides lie along three arbitrary sides of the given n-gon (Fig. 8).

Proof. First let us show that *any convex n-gon can be enclosed by a triangle or by a parallelogram whose sides lie along three or four sides respectively of the given n-gon.*

1°. For $n = 3$ our assertion needs no proof; for $n = 4$ it is almost obvious: either our quadrilateral is a parallelogram, or it has two nonparallel opposite sides which, together with the side adjacent to both of them, form a triangle which contains the quadrilateral inside itself (Fig. 9, *a*).

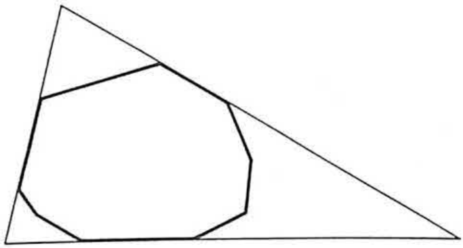

FIG. 8

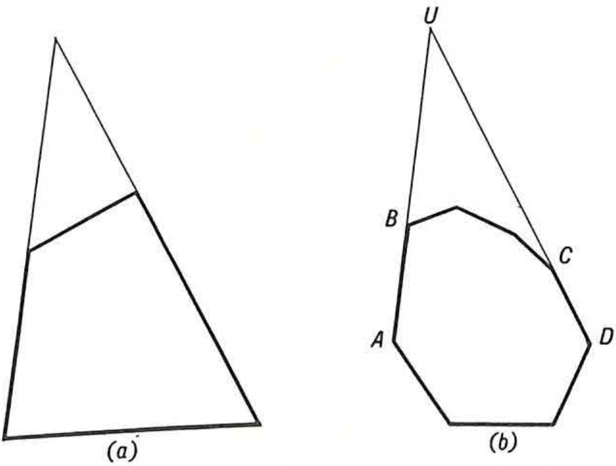

FIG. 9

2°. Suppose our assertion has been proved for all convex m-gons, where $m < n$. Consider a convex n-gon M (where $n \geqslant 5$). If AB is an arbitrary side of M, then M has another side which is not adjacent and not parallel to AB, since the total number of sides

in M, not counting AB or the sides not adjacent to it, is equal to $n - 3 \geqslant 5 - 3 = 2$ and the convex polygon M may have no more than one side different from AB and parallel to it. If we extend both AB, and the side CD which is not parallel to AB, until they intersect at the point U (we suppose that the points A, B, C, D, U are situated as in Fig. 9, *b*) and replace the polygonal (three-link) line BC enclosed inside the angle AUD by a two-link line BUC, then we shall obtain a polygon M_1 having fewer than n sides and containing the polygon M inside it, all the sides of M_1 being simultaneously the sides of M. By the induction assumption, there exists a triangle or parallelogram formed by the sides of M_1, which contains M_1 inside itself. But the sides of the triangle or parallelogram are simultaneously the sides of M, and it contains M inside itself, which proves the required assertion.

We now have to show how the assertion just proved (italicized on page 26) leads to the proposition stated in Example 12. If the polygon under consideration, which contains M inside itself, is a triangle, we have nothing more to prove; so we only have to consider the case when the polygon enclosing M is a parallelogram $PQRS$, and M itself is not a parallelogram (Fig. 10). Let P be a vertex of the parallelogram not belonging to the

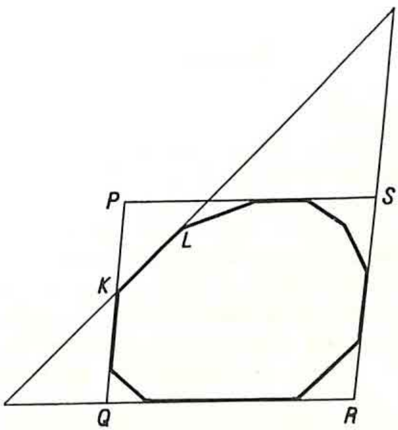

FIG. 10

vertices of M, and K the vertex of M nearest to P lying on the side PQ of the parallelogram; KL the side of the M emanating from K and directed inside the parallelogram. Since M is a convex polygon, it completely lies in the half-plane bounded by the

straight line KL and containing the vertices Q, R and S of the parallelogram. From this it follows that M is entirely enclosed inside a triangle with the sides KL, QR, RS (Fig. 10). This completes the proof of the proposition given in Example 12.

Example 12 is especially interesting, since the next proposition follows directly from it.

PROBLEM 9. Prove that any convex polygon which is not a parallelogram can be covered with three homothetic polygons smaller than the given one.

Hint. If a convex polygon $M \equiv A_1A_2 \ldots A_n$ is not a parallelogram, then it can be enclosed inside the triangle $T \equiv ABC$ formed by the sides $AB = A_1A_2$, $BC \equiv A_kA_{k+1}$, $\check{C}A \equiv A_lA_{l+1}$ belonging to the polygon (the identity $AB \equiv A_1A_2$ means here the coincidence of the *straight lines* AB and A_1A_2). Let O be an arbitrary interior point of M, and U, V and W points on the line segments A_1A_2, A_kA_{k+1}, A_lA_{l+1}, respectively. The segments OU, OV and OW cut M into three pieces which are covered by homothetic copies of M with the centres of similitude at the vertices of the triangle ABC and the ratios of magnification less than, but sufficiently close to, 1.

The result of Problem 9 was first proved in 1955 by a famous mathematician F. W. Levi. The same theorem was independently established in a quite different way by I. Ts. Gochberg and A. S. Markus in Kishinev (Moldavian SSR). Since the parallelogram Π can obviously be covered by only *four* copies of itself "reduced and positioned parallel to Π" (for, no such copy of Π can cover two different vertices of the parallelogram at once), we come to the following theorem which is sometimes called the Levi Theorem (or the Levi-Gochberg-Markus Theorem)*:

Let M be a convex polygon; the smallest number of "reduced copies" of M (in the sense explained above) *needed to cover M is* 3 *if M is not a parallelogram, and* 4 *if it is a parallelogram.*

This theorem gives a good idea of what "*combinatorial geometry*" is all about (it is a new direction in geometry stated in the 50's and 60's). In this subject the method of mathematical induction is widely used for proving various theorems. Combinatorial geometry deals with problems, connected with finite configurations

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* F. Levi, as did I. Gochberg and A. Markus, took not only a polygon but any *convex figure* Φ when considering coverings with "homothetic copies". However, it easily follows from the general properties of convex figures that the Levi theorem is valid for all (plane) convex figures if this theorem is fulfilled for convex polygons.

of points or figures (for further information see, for example, [3] and [9]). In these problems values are estimated connected with configurations of figures (or points) which are optimal in some sense. A clear understanding of the difficulties arising in this new field of geometry can be got by comparing the Levi theorem with the following hypothesis. This hypothesis is almost surely true, but many well-known mathematicians have failed to prove it:

The least number of "reduced copies" of a convex polynedron M (i. e. solids smaller than M, parallel and similar to M, the ratio of magnification being less than 1), *with which it is possible to cover the polyhedron* M, varies for various polyhedra between* 4 *and* 8 (for a tetrahedron, for example, this number is equal to 4, while it is equal to 8 for a cube). *The only polyhedra for which this number equals* 8 *are parallelepipeds.*

For more detailed information on the "Levi hypothesis" see [3].

A vast number of geometric theorems proved by the method of mathematical induction are connected with *geographical maps* or *planar graphs* (the term more oftenly used today).

Let there be given in a plane a network of lines, joining points $A_1, A_2, \ldots, A_p$ and having no other common points. We shall

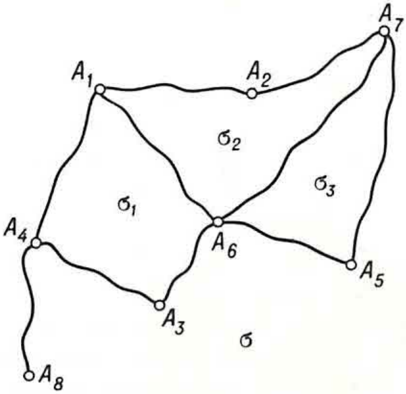


FIG. 11

say that the network is "in one piece", if from each of the points $A_1, A_2, \ldots, A_p$ it is possible to reach any other point by moving only along the lines of the network (the property of *connectedness*). We shall call such a network a *map*, the given points its *vertices*,

* It goes without saying that here, as in the two-dimensional case, instead of the polyhedron M, it is quite permissible to speak of an arbitrary *three-dimensional convex solid*.

the segments of the curves between two adjacent vertices the *borders* (or boundaries) of the map, and the parts of the plane into which it is divided by the borders, the *countries* of the map. Thus, in Fig. 11, the points A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8 are the vertices of the map, the curves A_1A_2, A_2A_7, A_1A_6, A_6A_7, A_4A_1, A_4A_3, A_3A_6, A_6A_5, A_5A_7, A_4A_8 – its borders, the regions σ_1, σ_2, σ_3 and the infinite exterior region σ – its countries.

EXAMPLE 13. *Euler's Theorem.* Let s denote the number of countries of an arbitrary map, l the number of its boundaries and p the number of vertices. Then

$$s + p = l + 2.$$

Proof. We proceed by induction on the number l of boundaries of the map.

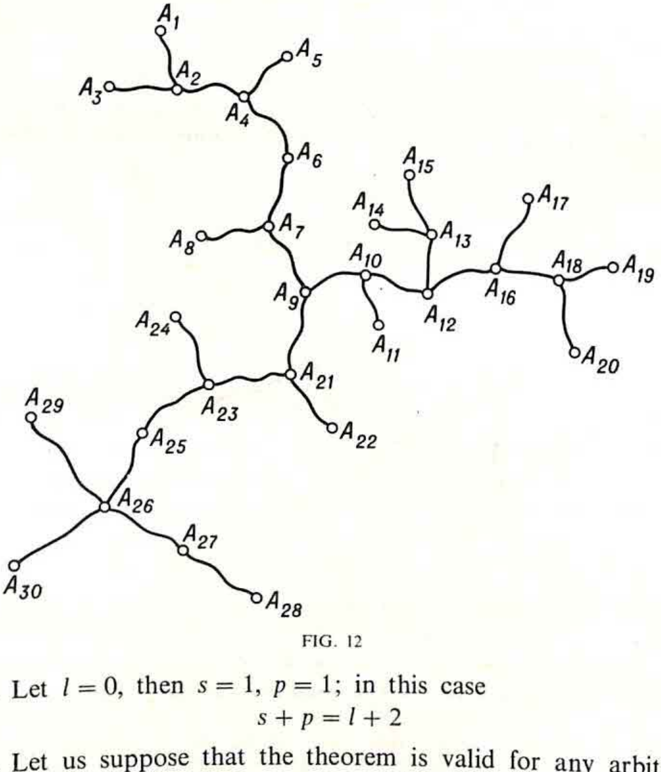


FIG. 12

1°. Let $l = 0$, then $s = 1$, $p = 1$; in this case

$$s + p = l + 2$$

2°. Let us suppose that the theorem is valid for any arbitrary map, having n boundaries, and consider a map with $l = n + 1$

boundaries, s countries and p vertices. There are two possible cases.

(a) For any two vertices of the map there exists a *single* path connecting them along the boundaries of the map (there is at least one such path due to the connectedness of the map). In this case the map does not contain a single closed contour and accordingly has the form depicted in Fig. 12; hence $s = 1$. Let us show that on such a map there can be found at least one vertex, belonging to only one boundary (say, the vertex A_1 in Fig. 12; we shall call such a vertex a *terminal* one). Now, let us take an arbitrary vertex of the map. If it is not terminal, then it serves as the end point of at least two boundaries. Let us move along one of these boundaries until its other end point is reached. If this vertex is also not terminal, then it serves as the end point of some other boundary; we proceed by the boundary to its second end point and so on. Since it is supposed that the map does not contain any closed contours, we shall never return to any vertex previously passed through, and since the number of vertices of the map is finite, we shall finally arrive at a terminal vertex. Deleting the vertex together with the single boundary having it as an end point, we obtain a new map, in which

$$l' = l - 1 = n, \quad s' = s = 1, \ p' = p - 1,$$

and, moreover, this new map remains connected. By the induction assumption

$$s' + p' = l' + 2,$$

whence

$$s + p = l + 2.$$

(b) There exist two vertices which can be joined by more than one path (Fig. 11). In this case there is a closed contour on the map passing through these vertices. Deleting one of the boundaries of this contour (without its vertices), we obtain a new connected map, in which

$$l' = l - 1 = n, \quad p' = p, \quad s' = s - 1.$$

By the induction assumption

$$s' + p' = l' + 2,$$

from which we get

$$s + p = l + 2.$$

EXAMPLE 14. Prove that if at each vertex of the map there meet no fewer than three boundaries (i. e. if the map does not contain such vertices as A_2, A_3, A_5, A_8 and such boundaries as A_4A_9 in Fig. 11), then there exists a country on the map having no more than five boundaries.

Solution. Since at each of the p vertices of the map there meet no fewer than three boundaries, $3p$ is not greater than twice the number of boundaries, $2l$ (twice the number, since each boundary joins two vertices); hence

$$p \leqslant \frac{2}{3} l. \tag{9}$$

We now suppose that each of the s countries of the map has no fewer than six boundaries; then $6s$ will not exceed twice the number of boundaries, $2l$ (twice the number, as each boundary separates two countries), hence

$$s \leqslant \frac{1}{3} l. \tag{10}$$

Inequalities (9) and (10) yield

$$s + p \leqslant \frac{1}{3} l + \frac{2}{3} l = l,$$

which contradicts Euler's theorem. Consequently, our assumption that each country has no fewer than six boundaries is untrue.

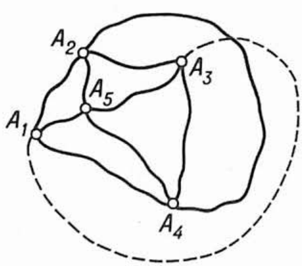


FIG. 13

PROBLEM 10. Given: five points in a plane. Prove that it is impossible to join all the points to each other in pairs by non-intersecting lines (Fig. 13).

Hint. Supposing that all these points are joined so that the conditions of the problem are satisfied, we arrive at a map having 5 vertices,

$\frac{5 \times 4}{2} = 10$ boundaries and, consequently, 7 countries (Euler's theorem!). The impossibility of such a map follows from an argument similar to that which led to inequality (10).

The reader can find other examples of the use of the Euler map theorem in [8].

PROBLEM 11. Prove that if p is the number of vertices, l the number of edges and s the number of faces of a convex polyhedron, then

$$s + p = l + 2$$

(*Euler's theorem on polyhedra*).

Hint. Place the polyhedron inside a sphere of a sufficiently large radius, and from the centre of the sphere (which may be assumed to lie inside the polyhedron) project all points of the polyhedron onto the surface of the sphere. Project the map so obtained on the surface of the sphere from any of its points, not belonging to any boundary, onto the plane tangential to the sphere at the diametrically opposite point (*stereographic projection*). Then apply Euler's theorem to the planar map thus obtained.

PROBLEM 12. Show that each polyhedron has either a triangular, a quadrangular, or a pentagonal face.

Hint. See Example 14.

PROBLEM 13. Show that there does not exist a polyhedron with seven edges.

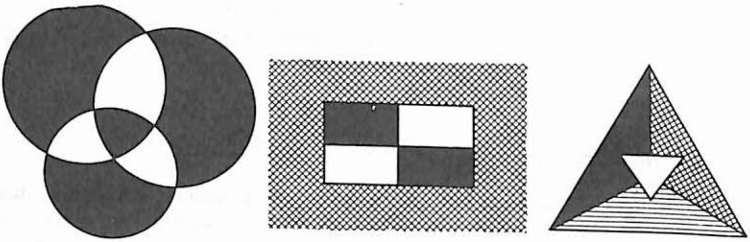
Hint. Use Euler's theorem.

The reader can find other examples of Euler's theorem on polyhedra in [21].

Map Colouring

Let a certain map be given in the plane. We shall say that it is *coloured properly* if each of its countries is coloured with a definite colour, so that any two adjacent countries (i. e. any two countries having a common boundary) are coloured with different colours. Any geographical map would serve as an example of a properly coloured map. Any map may be coloured in a correct way, for example, by assigning each of its countries its own special colour, though such a colouring is uneconomical. Naturally, there arises the question: what is the *least* number of colours which is necessary and sufficient to colour a given map properly? As is quite obvious, for example, the map shown in Fig. 14, *a* is colourable

with only two colours, whereas to colour the map illustrated in Fig. 14, *b* properly we need at least three colours. The map shown in Fig. 14, *c* can be properly coloured with no less than four colours. *No map* has been hitherto found whose proper colouring requires more than four colours. The well-known German mathematician, A. F. Möbius, was the first to pay attention to this fact about 140 years ago. Some ten years later this *four-colour-map*



(*a*) (*b*) (*c*)

FIG. 14

problem, as it was called eventually, was encountered by an Englishman Francis Guthrie in solving a practical problem of colouring the map of England in which the various counties had to have different colours. Francis Guthrie tried to solve this problem but failed. (In the English-language literature the four-colour-map problem is sometimes referred to as Guthrie's problem.) He wrote a letter to his younger brother Frederick, who was still at University College, London as a student of the mathematician Augustus De Morgan. Francis Guthrie pointed out to his brother Frederick that it seemed that every map drawn on a sheet of paper can be coloured with only four colours in such a way that countries sharing a common border have different colours. He asked if there was any way to prove this mathematically. Frederick did not know, and he asked De Morgan, the well-known professor at Cambridge University, who did not know either. But De Morgan made this problem popular among his colleagues. Thus, both the four-colour problem and the basic method of proof have deep roots in mathematics.

In 1878 the leading English mathematician, Arthur Cayley, unable to prove or disprove the four-colour conjecture, presented the problem to the London Mathematical Society. At the end of his report he invited everybody present to take part in solving this problem, thus "letting the genie out of the bottle". The four-

colour conjecture became a really famous problem. From that year on, for almost a century, every outstanding mathematician tried his best to solve this intractable problem, but no one was successful.

The first "solutions" of the four-colour problem appeared in 1880 soon after Cayley's report. Their authors were Alfred Bray Kempe, a barrister * and member of the society, and Peter Guthrie Tait. Not only Cayley, but other mathematicians, including the well-known Felix Klein, accepted these solutions. Gerhard Ringel, whose name we will meet again later on, pointed out that the unchallenged life of Kempe's argument for a decade may be cited as evidence that mathematicians of those days were no more prone to read each other's papers than they are today. But in 1890 the famous English mathematician, Percy John Heawood, carried out a thorough analysis of Kempe's and Tait's proof and showed that there was a fallacy in them. From that time on many other solutions of the problem appeared in mathematical books and journals, but all of them turned out to be erroneous (except for the last one which will be discussed below).**

The "proofs" suggested by Kempe and Tait, as well as many other erroneous solutions of the problem, were based on the *method of mathematical induction*; no other method or approach was even contemplated. Kempe's argument was most noteworthy (we shall consider it later on). In spite of the mistake made by this author, it contained a sensible and useful idea which played a leading role in further developments. P. J. Heawood discovered that from Kempe's reasoning it follows directly that any geographical map, containing no countries that are broken into a number of separate "pieces", can be properly coloured with *five* colours. (That was an irreproachably correct conclusion.) This result (see Example 18, below) was termed the *Five-Colour Theorem* (or the Kempe-Heawood Theorem). Thus, a disappointing gap was felt: on the one hand there are maps which cannot be properly coloured with *three* colours (Fig. 14, *c*), on the other hand, any map is colourable with *five* colours (the Kempe-Heawood Theorem). It is not difficult to find conditions which guarantee that a map can be properly coloured using *two* colours (even a *necessary*

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* A. Cayley for a long time was a professor at Cambridge University and a practising barrister at the London College of Barristers where he worked together with A. Kempe. The bar was more profitable than being a professor.

** For the history of the famous four-colour-map problem and its solution see the book [15] or the comparatively recent article [19]

and sufficient condition for this case, see Example 16), or with *three* colours (Example 17). The next hundred years saw many conditions necessary (or even necessary and sufficient, see Example 19) for a map to be able to be properly coloured with four colours. But the question whether such conditions are satisfied for any map, i. e. whether *any* geographical map drawn on a plane (or on a "globe", i. e. on a sphere, see below) can be properly coloured with *four* colours, remained unanswered for a long time.

Heawood dedicated all his life to the four-colour problem (and to his teaching). His contributions to the subject were considerable. Failing to solve the colour problem on the plane (or on the sphere *), Heawood started with maps on more complex surfaces. He began with the *torus* ("Saturn's rings" so to speak, see Fig. 15) and here he was unexpectedly successful. Heawood was clever enough to construct on the torus a map consisting of *seven* countries, each pair of countries having a common boundary (cf. Fig. 14, *c* illustrating a planar map containing *four* pairwise adjacent countries; let us mention that the whole problem arose

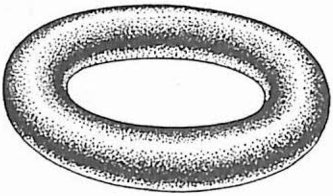

FIG. 15

with Möbius' proof of a simple theorem, that *no planar map can contain five countries, any two of which lie adjacent to one another*). It was obvious that "Heawood's map on the torus" cannot be

* The equivalence of the map colouring problems on a plane or a sphere may be established by the so-called stereographic projection. By *stereographic projection* we mean the projection of a sphere Σ from its "north pole" O onto a plane π which is tangential to Σ at its "south pole" Q. Clearly, if we have a map on the sphere Σ (we shall suppose that the point O does not lie on any boundary of the map), then the stereographic projection will project it to a map on the plane π, and neighbouring countries on the sphere Σ will be projected to neighbouring countries on the planar map. Hence for each map on the sphere there corresponds a map on the plane, which is colourable with the same number of colours (and vice versa). In other words the sphere and the plane have the same *chromatic number* (see below).

properly coloured with less than *seven* colours. On the other hand, Heawood proved that *each map on the torus is colourable with seven colours.* Thus, paradoxically as it may seem, on the torus, generated in a more complicated way than the plane or the

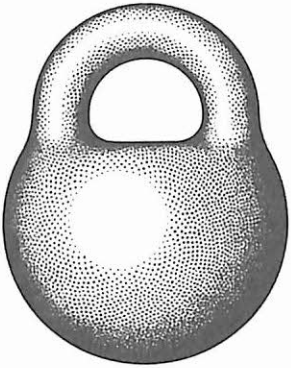

FIG. 16a

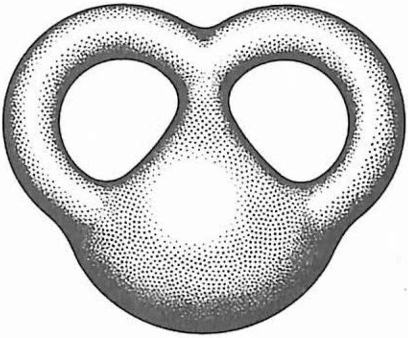

FIG. 16b

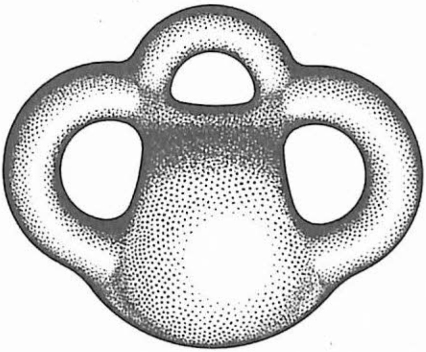

FIG. 16c

sphere, the map problem was completely solved: the minimal number of colours necessary for a proper colouring of any map on the torus is equal to *seven* (Heawood's Theorem)*. This result encouraged Heawood to continue his investigations into colouring maps on complex surfaces.

All closed surfaces (two-sided, i. e. having an "outer" side and an "inner" side) are constructed like spheres with a certain number of "handles" (or with a certain number of "holes"). Thus, the torus

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* For the proof of Heawood's Theorem see [10].

may be thought of as "a sphere with one hole", or "a sphere with one handle" (Fig. 16, *a*). Fig. 16, *b* shows a sphere with two holes constructed in the same way as "a sphere with two handles"; while Fig. 16, *c* depicts "a sphere with three holes" (or "three handles"), and so on*. Heawood was enthusiastic, thinking that he had found a simple formula giving the minimal number of colours necessary for a proper colouring of *any* map on the surface of a sphere with p "handles" (or "holes") – the so-called *chromatic number* Mp of a "sphere with p handles", $\sum_p$. There is his formula

$$M_p = \left[\frac{7 + \sqrt{1 + 48p}}{2}\right] \qquad (*)$$

in which the square brackets denote the *integral part* of a number ($[x]$ is the *greatest whole number not exceeding* x). For $p = 1$ the formula (*) yields $M_1 = 7$, i. e. the Heawood theorem; for $p = 0$ we obtain $M_0 = 4$, i. e. the result which was supposed to be true by Möbius; for $p = 2$, $M_2 = 8$ and so on. (By the way, Heawood held that he had proved the formula (*) only for *positive integers* $p > 0$).

Heawood published his formula in 1890 and called it the Map Colour Theorem, but in 1891 the German mathematician L. Heffter found some gaps in Heawood's argument, and from that year on (till 1968) formula (*) was called the *Heawood Conjecture* or *Heawood's hypothesis* (the problem was to prove it). However, as far as the torus was concerned (the case of $p = 1$) Heawood's reasoning was absolutely correct, but in the general case his argument only yielded the inequality $M_p \leqslant H_p$, where H_p is Heawood's number (i. e. the number written on the right-hand side of formula (*)). Heffter established that "Heawood's hypothesis" is true for the first few values of p.

In the decades which followed Heffter's contribution there were no further successes. Finally, in the 50's Heawood's conjecture was decisively attacked by the talented mathematician G. Ringel, who was joined in the 60's by the energetic American J. W. T. Youngs. About twenty years of hard work spent by the two talented mathematicians was crowned with a great success – in 1968 their joint efforts led to the solution of all the twelve cases into which the proof of Heawood's hypothesis was divided. Beautiful combinatorial methods were developed in order to prove the Heawood formula. They also obtained stronger results on *nonorientable* surfaces (see [4] or [6]). All the results obtained by G. Ringel

* See [6]

and J. W. T. Youngs can be found in the book "Map Colour Theorem" written by Gerhard Ringel after the sudden death of Youngs, and dedicated to Professor Youngs.

With the formula (*) proved for $p > 0$, only one case, $p = 0$, remained open, i. e. the initial Four-Colour Problem (Guthrie's problem).

With these remarks we put aside for a while our brief historical survey and pass over to examples and problems on map colouring.

We shall assume from now on that the map has no nondividing boundaries, i. e. boundaries on both sides of which the same country is found (as the boundary A_4A_8 in Fig. 11), since otherwise the statement of the problem concerning a proper colouring is meaningless. We shall also assume that the map contains no vertex in which only two boundaries meet (as, for instance, the vertex A_2 in Fig. 11), since such a vertex would be superfluous. In other words, we shall consider only those maps at each vertex of which at least three boundaries come together, i. e. maps satisfying the condition of Example 14. (The result of this example will be repeatedly used later on.) It will be convenient for us to suppose that the map has only one infinite region, i. e. it has no boundaries "going off to infinity". It can be shown that the last condition is not essential, since leaving it out does not change any of the further results.

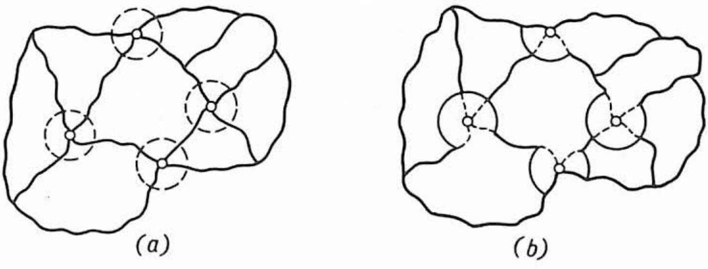


FIG. 17

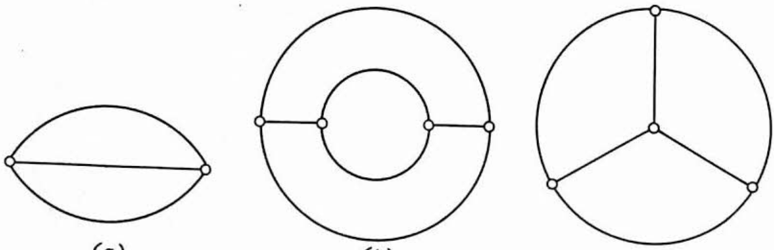
A map is called *normal* if none of its countries enclose other countries and if no more than three countries meet at any vertex (the term itself as well as the below described method of reducing any map to a normal one, were introduced by A. B. Kempe in 1879). Let there be given an arbitrary map S (Fig. 17, a). By describing a sufficiently small circle around each vertex of this map at which more than three boundaries meet, and adding

it to one of the countries surrounding this vertex, we obtain a normal map S' having the same number of countries (Fig. 17, *b*). Any proper colouring of the map S' readily yields a proper colouring of the map S using the same number of colours, and vice versa. So, in solving the problem of the proper colouring of maps we shall often confine ourselves to considering normal maps.

Let us now find out how simplest normal maps are constructed*. Let p be the number of vertices, l the number of boundaries and s the number of countries of a normal map; then $2l = 3p$, whence $p = \frac{2}{3} l$. On the other hand, by Euler's theorem, $s + p = l + 2$; hence

$$s = (l - p) + 2 = \frac{l}{3} + 2$$

and, consequently, $s \geqslant 2$. But for $s = 2$ we find that $l = 0$. Obviously, such a map does not exist. Putting $s = 3$, we get $l = 3$ and $p = 2$. This simplest normal map has the form shown in Fig. 18, *a*. For $s = 4$ we get $l = 6$ and $p = 4$. Let us show



(*a*) (*b*) (*c*)

FIG. 18

that in this case the map has the form illustrated in Fig. 18, *b* or *c*. Let us denote the number of map bigons by k_2, the number of its triangles by k_3, and the number of quadrilaterals by k_4 (since $p = 4$, it is impossible for a map to have countries whose

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* Here and elsewhere we are not going to distinguish between 'equally constructed' maps (as illustrated in Figs. 14, *c* and 26, *a*), whose countries and boundaries can be numbered so that in both maps equally numbered countries are separated by equally numbered boundaries.

number of vertices exceeds four). Then

$$k_2 + k_3 + k_4 = s = 4$$

and

$$2k_2 + 3k_3 + 4k_4 = 2l = 12$$

(see page 32). As is obvious from the last equality, k_3 is an even number. The sum $k_2 + k_3 + k_4$ being equal to 4 (accurate to the order of the terms) may have the form $2 + 2 + 0$, $2 + 1 + 1$, $3 + 1 + 0$, $4 + 0 + 0$. Let us consider each of these cases.

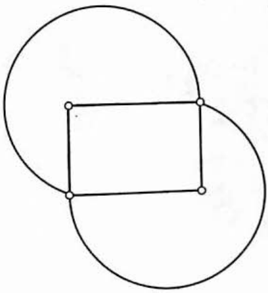

FIG. 19

If two values of k are each equal to 2, and the third is equal to zero, then for $k_2 = 2$, $k_3 = 2$, $k_4 = 0$ the sum $2k_2 + 3k_3 + 4k_4 = 10 < 12$; for $k_2 = 2$, $k_3 = 0$, $k_4 = 2$ the sum $2k_2 + 3k_3 + 4k_4 = 2l = 12$. This case corresponds to the map shown in Fig. 18, *b*. For $k_2 = 0$, $k_3 = 2$, $k_4 = 2$ the sum $2k_2 + 3k_3 + 4k_4 = 14 > 12$. If one of the values is equal to 2 and the other two each equal to 1, then only for $k_2 = 1$, $k_3 = 2$, $k_4 = 1$ is the sum $2k_2 + 3k_3 + 4k_4$ equal to 12. Such a map exists, but is not normal (see Fig. 19).

If one of the values of k is equal to 3 and one to 1, then k_3 must equal zero, since k_3 is even. In this case $2k_2 + 4k_4 \neq 12$.

Finally, if one of the values of k is 4, and the other two are zero, then only when $k_2 = k_4 = 0$, $k_3 = 4$ is the sum $2k_2 + 3k_3 + 4k_4$ equal to $2l = 12$. The corresponding map has the form shown in Fig. 18, *c*.

EXAMPLE 15. Given: n circles in a plane, prove that the circles arranged in any way form a map colourable with two colours.

Solution. 1°. For $n = 1$ the assertion is obvious.

2°. Suppose that our assertion is true for any map formed by n circles, and let there be given $n+1$ circles in a plane. By removing one of the circles, we obtain a map which, by the hypothesis, can be properly coloured by means of two colours, say, black and white (Fig. 20, *a*). Then reintroduce the circle that was removed, and on one side of it (for instance, inside the circle), change the colour of each region to the opposite colour (i. e. white instead of black, and vice versa). As can be clearly seen, we obtain in this way a map colourable with two colours (Fig. 20, *b*).

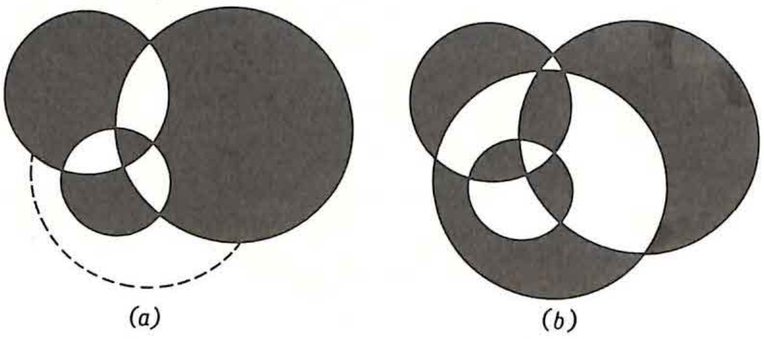

FIG. 20

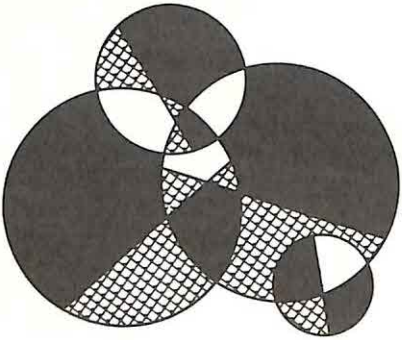

FIG. 21

PROBLEM 14. Given: n circles in a plane, each with a chord drawn across it. Prove that the map so formed is colourable with three colours (Fig. 21).

Hint. Suppose that the map formed by n circles each having one chord be properly coloured with three colours α, β, γ.

Draw an $(n+1)$-th circle and change the colours of the countries lying inside this circle on one side of the chord according to the scheme $\alpha \to \beta$, $\beta \to \gamma$, $\gamma \to \alpha$, and the colours of the countries situated inside the circle on the opposite side of the chord according to the scheme $\alpha \to \gamma$, $\beta \to \alpha$, $\gamma \to \beta$.

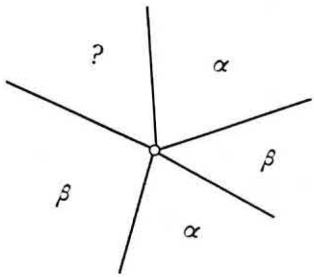

FIG. 22

EXAMPLE 16. (*Two-Colour Theorem*). For a map to be colourable with two colours it is necessary and sufficient that an even number of boundaries meet at each of its vertices.

Solution. The *necessity* of this condition is obvious, since if an odd number of boundaries meet at some vertex, then the countries surrounding this vertex cannot be properly coloured using only two colours (see Fig. 22).

To prove the *sufficiency* of the condition let us carry out an induction on the number of map boundaries.

FIG. 23

1°. For a map with two boundaries the proposition is obvious (Fig. 23).

2°. Suppose that the theorem is true for any map where an even number of boundaries meet at each vertex and whose total number of boundaries does not exceed n. Let there be given

a map S having $n+1$ boundaries and satisfying the same condition. Beginning with an arbitrary vertex A of the map S let us start moving along its boundaries in some arbitrary direction. Since the number of map vertices is finite we shall eventually return to one of the vertices already passed through (the map has no terminal vertices, since it does not have any nonseparating

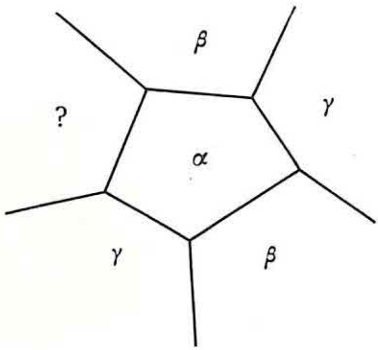

FIG. 24

boundaries) and so we are able to select a closed contour formed by map boundaries which does not intersect itself. Removing this contour, we get a map S' with a smaller number of boundaries and an even number of boundaries meeting at each vertex (because an even number of boundaries – 0 or 2 – was removed at each vertex of the map S). By the induction assumption, the map S' can be properly coloured with two colours.

Restoring the contour that we removed and changing all the colours on one of its sides (for instance, inside), we obtain a proper colouring for the map S.

EXAMPLE 17. (*Three-Colour Theorem*). For a normal map to be colourable with three colours it is necessary and sufficient that each of its countries have an even number of boundaries.

Proof. The *necessity* of the condition is obvious, since if there is a country σ on the map with an odd number of boundaries, then it is impossible to properly colour σ and the bordering countries using three colours (see Fig. 24).

To prove the *sufficiency* of the condition let us carry out an induction on the number n of countries on the map.

1°. For a normal map containing three countries (see Fig. 18)

our proposition is obvious. A normal map consisting of four countries (shown in Fig. 18, *b*) is also colourable with three colours. To do this it is sufficient to colour the "interior" country with the same colour, which was used for the "exterior" region. The normal map shown in Fig. 18, *c* does not satisfy the condition of evenness of the number of boundaries of each country. Thus, each normal map containing 3 or 4 countries each having an even number of boundaries is colourable with three colours.

2°. Suppose that the theorem is true for any normal map in which every country has an even number of boundaries and the total number of countries is equal to $n-1$ or n. Consider a normal map S satisfying the same condition and having $n+1$ countries. As follows from Example 14, we can find on the map S a country σ having not more than five boundaries. In our case the country σ will have either two, or four boundaries. Let us consider each case.

A. The country σ *has two boundaries*. Let A and B be its vertices, and σ_1 and σ_2 its neighbouring countries (Fig. 25). Deleting the boundary between the countries σ and σ_1, we get a map S' which is also normal, since the points A and B are no longer vertices (we stipulated that the map has no superfluous boundaries), and the number of boundaries as before come together at the remaining vertices. Each country of the map S' also has an even number of boundaries, since the number of boundaries of each

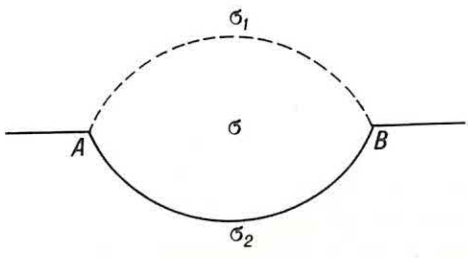

FIG. 25

of the countries σ_1 and σ_2 is reduced by 2, and the number of boundaries of the remaining countries is unchanged. Since the map S' has n countries, by the induction assumption it is colourable with the three colours α, β, γ. Let the countries $\sigma_1' = \sigma_1 + \sigma$ and $\sigma_2' = \sigma_2$ be coloured with α and β, respectively. Restoring the country σ and colouring it with γ, we get a proper colouring for the map S.

B. The country σ *has four boundaries*. It may happen that any two of the countries adjacent to σ on opposite sides, border upon each other or even coincide (Fig. 26, *a* or 18, *b*). But in this case the other two countries bordering on σ cannot have common boundaries, or coincide. Let the countries σ_2 and σ_4 (Fig. 26, *b*)

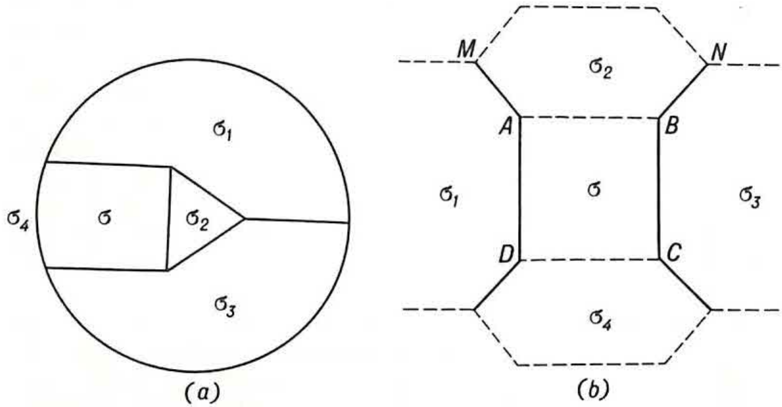

FIG. 26

be like this. Let us join the countries σ_2 and σ_4 to σ by deleting the boundaries AB and CD. In this way we obtain the map S' which is, clearly, also normal, each of its countries having an even number of boundaries. In fact, if the numbers of the boundaries of the countries σ_1, σ_2, σ_3 and σ_4 are equal to $2k_1, 2k_2, 2k_3$ and $2k_4$, respectively, then the country $\sigma' = \sigma + \sigma_2 + \sigma_4$ will have $2k_2 + 2k_4 - 4$ boundaries, the country $\sigma'_1 = \sigma_1$ will have $2k_1 - 2$ boundaries and the country $\sigma'_3 = \sigma_3$ will have $2k_3 - 2$ boundaries, while each of the remaining countries retains its number of boundaries. (If the countries σ_1 and σ_3 coincide, then the total number of boundaries of this country on the map S' is by 4 less than on the map S.)

Since the map S' has $n - 1$ countries, by the induction assumption, it is colourable with the three colours α, β, γ. Let us show that in this case the countries σ'_1 and σ'_3 will be coloured with one and the same colour (this assertion is obvious if σ'_3 coincides with σ'_1). Indeed, suppose the country σ' be coloured with α and σ'_1 with β. Since on the region MN an odd number of countries $(2k_2 - 3)$ are adjacent to σ', their colours must obviously alternate in the succession $\gamma, \beta, \gamma, \beta, \ldots, \gamma$, so the country σ'_3 will be coloured with the colour β. Restoring the country σ and colouring it with γ, we get proper colouration for map S.

EXAMPLE 18. (*Five-Colour Theorem*). Any normal map is colourable with five colours.

Proof. 1°. For a map containing not more than five countries the assertion is obvious.

2°. Suppose that the theorem is true for any normal map containing $n-1$ or n countries and consider a map S consisting of $n+1$ countries. As shown in Example 14, the map S contains at least one country σ, the number of whose boundaries does not exceed five. All the possible cases are considered below.

(a) The country σ *has two boundaries* (see Fig. 25). Let σ_1 and σ_2 be adjacent to σ. By joining σ_1 to σ we get a normal map S' with the number of countries equal to n.

By the induction assumption, the map S' is colourable with five colours. With this colouring the countries $\sigma_1' = \sigma + \sigma_1$ and $\sigma_2' = \sigma_2$ will turn out to be coloured with two of these colours. Restoring the country σ, we can colour it with one of the three remaining colours.

(b) The country σ *has three boundaries* (Fig. 27, *a*). Join σ_1 to σ. When colouring the map S' so obtained with five colours, we shall be able to colour the country σ with one of the two colours not used in colouring the countries $\sigma_1' = \sigma + \sigma_1$, $\sigma_2' = \sigma_2$, and $\sigma_3' = \sigma_3$.

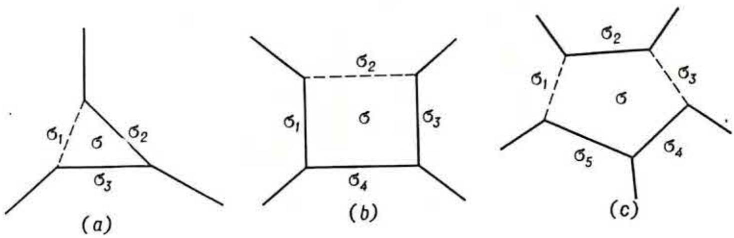

FIG. 27

(c) The country σ *has four boundaries* (Fig. 27, *b*). Among the countries adjacent to σ there can be found two countries which do not coincide (see Example 17). By joining one of these two countries, say σ_2, to σ, we get a map S' containing n countries which is colourable with five colours (by the induction assumption). In such a colouring the countries $\sigma_1' = \sigma_1$, $\sigma_2' = \sigma_2 + \sigma$, $\sigma_3' = \sigma_3$ and $\sigma_4' = \sigma_4$ will be given four of the five possible colours (or fewer, if σ_1' and σ_3' coincide or are coloured with the same colour). Restoring the country σ we can colour it with the remaining (fifth) colour.

(d) The country σ *has five boundaries* (Fig. 27, *c*). As in Example 17, we can find two nonbordering and noncoincident countries among those adjacent to σ. Let us denote these countries by σ_1 and σ_3. By joining them to σ we obtain a normal map S' having $n-1$ countries. By the induction assumption, the map S' is colourable with five colours. In such a colouring of the countries $\sigma'_1 = \sigma_1 + \sigma + \sigma_3$, $\sigma'_2 = \sigma_2$, $\sigma'_4 = \sigma_4$ and $\sigma'_5 = \sigma_5$ we must use four of the five colours. Restoring the country σ, we are able to colour it with the fifth colour.

Sometimes we shall colour not only the countries, but the *boundaries* of the map as well, numbering the colours used *1*, *2*, *3*, ... If all the boundaries meeting in one and the same vertex have different numbers, then such numbering will be called a *proper* numbering (see, for instance, Fig. 28). It is an interesting fact that the problem of a proper colouring of countries of a map is closely linked with the problem of a proper "colouring" of its boundaries, as is clearly seen from the following example.

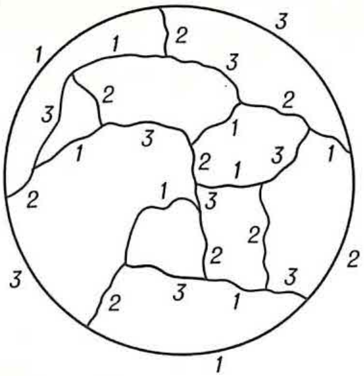

FIG. 28

EXAMPLE 19. (*Tait's Theorem*)*. A normal map is colourable with four colours if and only if its boundaries are numberable with three numbers.

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* This theorem was "roughly" (i. e. omitting some essential details) proven by P. G. Tait, the author of one of the first "proofs" of the Four-Colour Theorem (cf. [10]). In 1940 this theorem was independently reproven by V. V. Volynskii (1923–43), then a student of Moscow University (who was not aware of Tait's work) (Volynskii perished at the front during the war). This is why in the original Russian text Example 19 is called "Volynskii's Theorem".

Solution. A. If a normal map is colourable with four colours, then its boundaries are numberable with three numbers.

Let a normal map S be properly coloured with the four colours α, β, γ, δ. Let us denote the boundaries between the countries coloured with α and β or γ and δ by the number *1*, those between the countries coloured with α and γ or β and δ by *2*, and the boundaries between the countries coloured with α

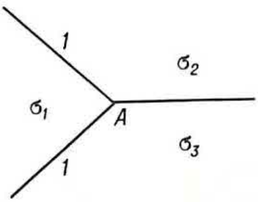

FIG. 29

and δ or β and γ by *3*. This notation yields a proper numbering of the boundaries: since if two boundaries numbered with one and the same figure (for instance, with the number *1* in Fig. 29) meet at some vertex A, then the countries σ_2 and σ_3 separated from the country σ_1 by identically numbered boundaries must be coloured with the same colour (if σ_1 in our example is coloured with α, then σ_2 and σ_3 must be coloured with β). But it is impossible since σ_2 and σ_3 are adjacent to each other.

B. If the boundaries of a normal map can be properly numbered with three figures, then its countries are colourable with four colours. We are going to prove this by carrying out an induction on the number n of countries on the map.

1°. The boundaries of the simplest normal map consisting of three countries (see Fig. 18, *a*) can be numbered by the figures *1*, *2*, *3* uniquely (to within permutation of these figures). Let us colour this map as shown in Fig. 30, *a*. In this case the boundary between the countries coloured with α and β will have number *1*, the boundary between the countries coloured with β and δ number *2* and the one between countries coloured with α and δ number *3*.

A proper colouring of a map S with the four colours α, β, γ, δ, in which the boundaries between the colours α and β, and between γ and δ are numbered with the figure *1*, the boundaries between the colours α and γ, and between β and δ with the figure *2* and those between the colours α and δ, and between β and γ with the number *3* will be called *permissible*. We have shown

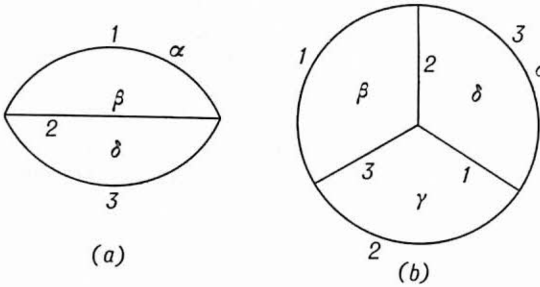

FIG. 30

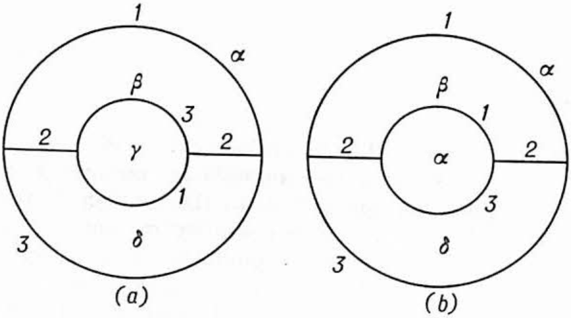

FIG. 31

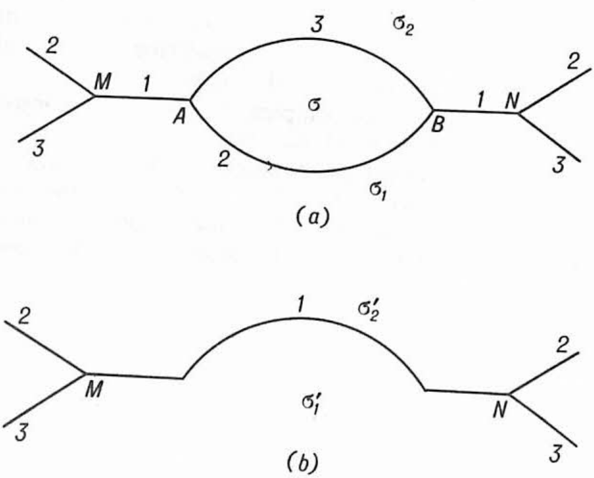

FIG. 32

that the simplest normal map consisting of three countries can be properly coloured with four colours in a permissible way. Let us now show that the same statement is valid for a normal map consisting of four countries (Fig. 18, *b* and *c*). The boundaries of the map shown in Fig. 18, *c* can be properly numbered in a unique way with the numbers *1*, *2*, *3* (Fig. 30, *b*) (to within permutation of these numbers). The colouring of this map illustrated in Fig. 30, *b* will be permissible. The map depicted in Fig. 18, *b* allows two essentially different ways of numbering its boundaries (Fig. 31, *a* and *b*). Colourings (of these maps) indicated in Fig. 31, *a* and *b* are also permissible.

2°. Let us assume that any normal map whose boundaries may be properly numbered with three figures, and which consists of $n-1$ or n countries, is colourable with four colours in a permissible way, and consider a normal map S having $n+1$ countries whose boundaries may also be properly numbered with three figures. As was stated in Example 14, we can find on the map S a country σ having not more than five boundaries. Consider all possible cases.

(a) The country σ *has two boundaries*. The unique (to within permutation of the figures) numbering of the boundaries in the

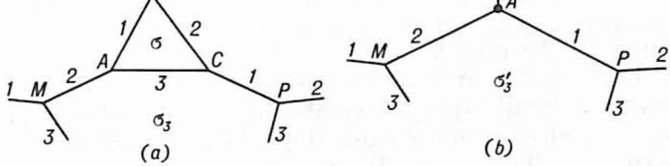

FIG. 33

neighbourhood of σ is shown in Fig. 32, *a*. Let us adjoin the country σ_1 to σ. Denote the new boundary MN separating the countries $\sigma_1' = \sigma_1 + \sigma$ and $\sigma_2' = \sigma_2$ (Fig. 32, *b*) by the number *1*, leaving the rest of the boundaries with their old numbers. The map S' so obtained is normal and its boundaries can be properly numbered with three numbers. Since the map S' contains n countries, it is colourable with four colours. If the country σ_1'

is coloured with α, then the country σ_2' will be coloured with β. Restoring the country σ and colouring it with γ, we get a permissible colouring for the map S with four colours.

(b) The country σ *has three boundaries*. The only possible numbering of the boundaries in the neighbourhood of σ is given in Fig. 33, *a*. Let us imagine that the map S is drawn on a rubber film. Contract the country σ to a point. As a result, the boundaries AB, BC and AC will disappear, and the vertices

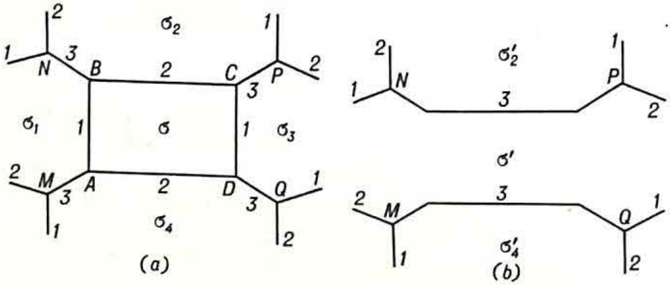

FIG. 34

A, B, C will merge into one: $A = B = C = A'$ (Fig. 33, *b*). Without changing the numbering of MA', NA', PA' (formerly MA, NB, PC) or the remaining boundaries, we get a normal map S' with its boundaries properly numbered. Since the number of the countries forming the map S' is equal to n, it is colourable with four colours. If in the map the country σ_1' is coloured with α, the country σ_2' will be coloured with δ and σ_3' with γ. Restoring the country σ and colouring it with β we get a permissible colouring for the map S.

(c) The country σ *has four boundaries*. In this case two essentially different ways of numbering the boundaries in the neighbourhood of σ are possible (Figs. 34, *a* and 35, *a*). Consider the first way (Fig. 34, *a*). There can be found two countries adjacent to σ which have no common boundaries (see Example 17). Since both pairs of opposite countries σ_1, σ_3 and σ_2, σ_4 are equivalent as far as the boundary numbering is concerned, we can assume that the countries σ_1 and σ_3 have no common boundaries. We now join the countries σ_1 and σ_3 to σ, assigning the number *3* to the new boundaries NP and MQ (Fig. 34, *b*). The map S' so obtained will be a normal one, with the boundaries properly numbered. Since this map contains $n - 1$ countries, it is colourable with four colours. If the country $\sigma' = \sigma_1 + \sigma_3 + \sigma$ is coloured

with α, then the countries $\sigma'_2 = \sigma_2$ and $\sigma'_4 = \sigma_4$ will be coloured with δ. Restoring the country σ, we colour it with β.

In the second case (Fig. 35, *a*) if σ_1 and σ_3 are the countries having no common boundaries, then we can reason in a similar way. But in this case the new boundary NP is given the number *3*, as before, and the boundary MQ the number *2* (Fig. 35, *b*). The country $\sigma'_4 = \sigma_4$ will be coloured with γ. Restoring the country σ, we colour it with β as before.

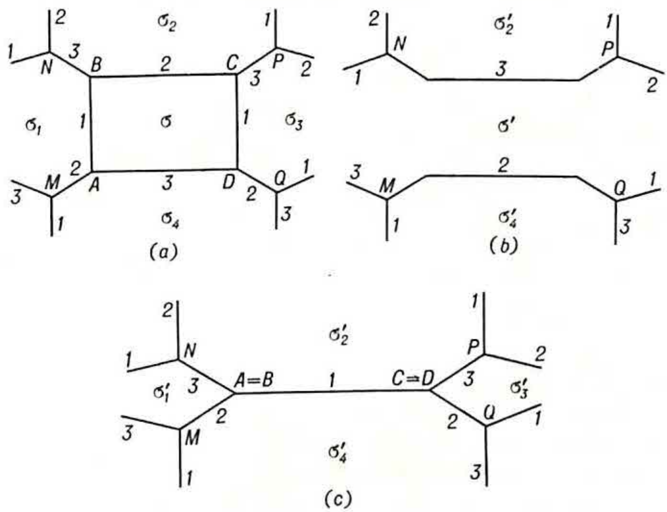

FIG. 35

Finally, suppose that the countries σ_2 and σ_4 have no common boundaries. Contract the quadrilateral $ABCD$ into a line segment so as to bring the point A into coincidence with the point B, and the point C with D. As a result, the boundary BC will merge with AD. Leaving he numbering of the boundaries MA, NB, PC and QD unchanged, we denote the new boundary $BC = AD$ by number *1* (Fig. 35, *c*). The map S' so obtained is a normal one with properly numbered boundaries. Since the number of its countries is equal to n, it is colourable with four colours. In this case if the country σ'_1 is coloured with α, then the country σ'_2 will be coloured with δ, σ'_3 with α and σ'_4 with γ. Restoring the country σ, we colour it with β.

(d) The country σ *has five boundaries*. In this case only one method of numbering the boundaries in the neighbourhood of σ

is possible (to within permutation of the numbers *1*, *2*, *3*) (Fig. 36, *a*). Consider first the case when the country σ_5 neither coincides nor has common boundaries with σ_2 and σ_3. Let us join the country σ_5 to σ, label the new boundary MB by the number *2*, the new boundary RD by *1*, give the boundary BC a new number *1* and the boundary CD a new number *2*. Then, as a result, we get a normal map S' (Fig. 36, *b*) with properly numbered boundaries. Since the number of its countries is equal to n, it is colourable with four colours. If the country $\sigma' =$

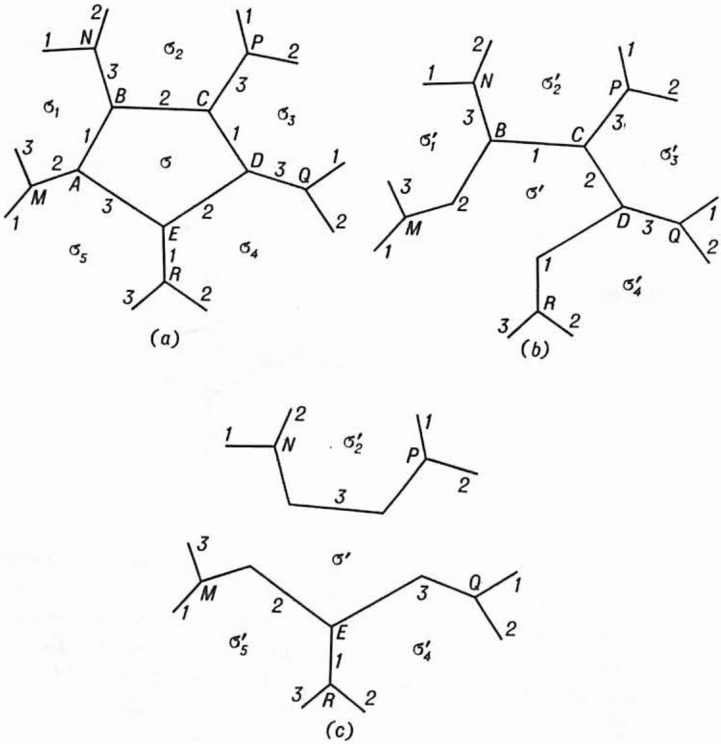

FIG. 36

$= \sigma + \sigma_5$ is coloured with α, then the countries σ'_2 and σ'_4 will be coloured with β, and the countries σ'_1 and σ'_3 with γ. Restoring the country σ, we can colour it with δ.

If the country σ_5 borders upon, or coincides with σ_2, then the countries σ_1 and σ_3 neither have a border in common, nor

coincide. If the country σ_5 coincides with or borders upon σ_3, then the countries σ_2 and σ_4 cannot coincide or be neighbours. Since the last two cases are equivalent as far as the boundary numbers are concerned, it is sufficient to consider the case when the countries σ_1 and σ_3 neither have common boundaries, nor coincide. Join them both to σ, and label the new boundary NP by *3*, the new boundary ME by *2* and the new boundary EQ by *3*. We then get a normal map S' (Fig. 36, *c*) with properly numbered boundaries. Since the number of its countries is equal to $n-1$, it is colourable with four colours. If the country $\sigma' = \sigma + \sigma_1 + \sigma_3$ is coloured with α, then the countries $\sigma'_2 = \sigma_2$ and $\sigma'_4 = \sigma_4$ will be coloured with δ, and the country $\sigma'_5 = \sigma_5$ with γ. Restoring the country σ, we colour it with β.

Since any normal map is colourable with four colours (Appel-Haken's Theorem, see page 62), the boundaries of each normal map are colourable with three colours (or numberable with three numbers). However, the complete proof of this fact is based on rather cumbersome methods. Therefore, we confine ourselves here to the following results which are not so strong but can be proved more easily.

EXAMPLE 20. The boundaries of every normal map can be numbered using four numbers. In the application to normal geographical maps this result, as has been already stated, can be made stronger, since, as follows directly from Tait's Theorem (see Example 19) and Appel-Haken's Theorem (see page 62), the boundaries of such a map are colourable with *three* colours. But we are going to prove this proposition for any network of lines in the plane (the map may even be disconnected, i. e. it may be in several separate pieces at each of whose vertices there come together not more than three lines ("boundaries")). For this it is better to speak not of a map but of a planar *graph* (see, for instance, [14]) formed by line-segments (the *arcs* or *edges* of the graph) meeting at certain points (the *vertices* of the graph). Example 20 gives an estimate of the number of colours needed for a proper colouration of the map boundaries (or graph arcs), which turns out to be *correct*. For instance, the four boundaries of the map illustrated in Fig. 37 obviously cannot be properly coloured with fewer than four colours (this map has a "fictitious" vertex A whose existence we eliminate when we discuss the problem of colouring the countries of a geographical map).

Let us note here that the problem of the proper colouring

of lines which form some complicated network (considered in Example 20 and in Problem 15, below), has some practical applications, particularly in electrical engineering. In order not to mix up connecting leads (or wires) in complex electric circuits it is convenient to use coloured wires (or leads whose ends are marked with coloured tags). The terminals of the instruments

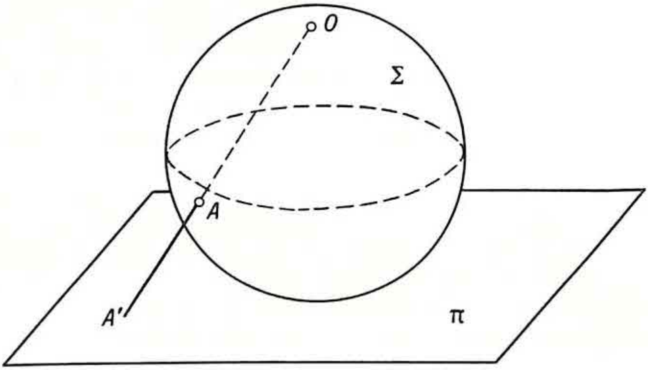

FIG. 37

are coloured correspondingly with the same colours as the conductors. It goes without saying that two leads going to a given terminal do not have to have the same colour. Thus, terminals of the electrical instruments here play the role of map vertices (or graph vertices), while the conductors play the role of map boundaries (graph arcs).

Proof (cf. [28], solution of Problem 115, *b*). We shall use the method of mathematical induction on the number n of map vertices.

1°. For $n = 2$ the assertion is obvious.

2°. Let us assume that our assertion is valid for any map, at each vertex of which there come together not more than three boundaries, and the number of vertices of which is equal to n. Consider the map S satisfying the same condition and having $n+1$ vertices. Removing one of these vertices, say A_0, together with the boundaries belonging to it, we get a map S', containing n vertices for which at each vertex there come together not more than three boundaries. By virtue of the induction assumption the boundaries of this map can be properly numbered with four numbers: *1*, *2*, *3*, *4*. Let us restore the vertex A_0 with its boundaries. Consider the three possible cases.

(a) The vertex A_0 is joined (by one, two or three boundaries) to only one vertex A_1 of the map S' (Fig. 38, *a*, *b*, *c*). In this case the numbering of the boundaries of the map S' is readily extended to a proper numbering of the map S.

(b) The vertex A_0 is joined to two vertices A_1 and A_2 of the map S', with one of which it may be connected with two boundaries (Fig. 39, *a* and *b*). It is easy to verify that in all cases the numbering of the boundaries of the map S' can be extended to a proper numbering of the boundaries of the map S.

(c) The vertex A_0 is connected with three vertices A_1, A_2, A_3 of the map S' (Fig. 40). The least favourable case is when at each of the vertices A_1, A_2, A_3 of the map S' two boundaries come together. In this case for each of these boundaries A_0A_1, A_0A_2, A_0A_3 we will have two possible numbers, from which we are unable to choose three different numbers, only when the three pairs are equal, i. e. when the three pairs of the boundaries of the map S' passing through the vertices A_1, A_2, A_3 are assigned the same numbers, say *1* and *2*. Select then, on the map S', a contour of maximum length beginning at the vertex A_1 and consisting of boundaries numbered alternately *1* and *3* (such a contour may consist only of one boundary and may also end at one of the vertices A_2 or A_3). This contour cannot have self-intersections, since, by assumption, the boundaries of the map S' are properly numbered. Let us change the numbers of the boundaries of this contour substituting *3* for *1* and vice versa. As is obvious, the map S' will still be properly numbered and in the new numbering the three pairs of boundaries passing through the vertices A_1, A_2 and A_3 of the map S' will not be numbered equally. And in this case the proper numbering of the boundaries of the map S' can readily be continued to form a proper numbering of the boundaries of the map S.

Example 20 is a particular case of the following (more general) assertion:

PROBLEM 15 (*Shannon's Theorem*)*. Given a map on the plane. At each vertex of the map there meet not more than k boundaries (i. e. the "map" can be any network of lines, any graph on the plane with not more than k "edges" or "arcs" meeting at each of its vertices). Prove that the boundaries of the map (the edges of the graph) can be properly coloured with

* Claude Shannon, an outstanding American mathematician, the founder of information theory.

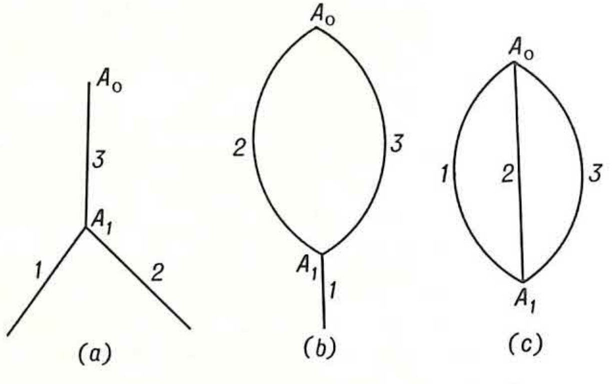

FIG. 38

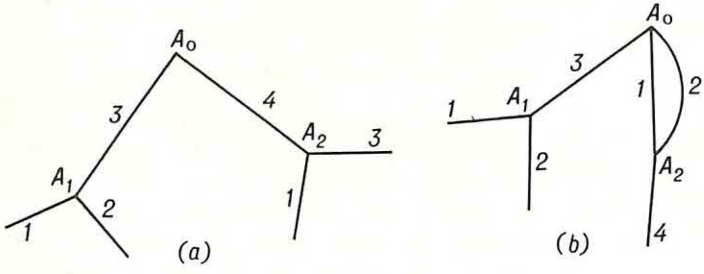

FIG. 39

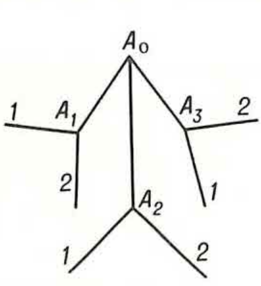

FIG. 40

$[(^3/_2)k]$ colours (where the brackets denote the *integer* part), this estimate being exact (i. e. for each k there exists a planar graph at each vertex of which k edges come together, such that its edges cannot be properly coloured with $[(^3/_2)k] - 1$ colours).

Hint. The Shannon theorem (see [12], [20]) is proved analogously to solving Example 20 which is a particular case of this theorem (since for $k = 3$ we have $[(^3/_2)k] = [^9/_2] = 4$).

The Tait theorem (Example 19) establishes an indirect connection between problems concerning proper colourings of countries and boundaries of a geographical map. The connections between problems concerning proper colourings of countries and vertices is much more direct. By a proper colouring of the vertices of a map (or of an arbitrary planar graph, see Fig. 41) we mean a colouring in which a certain colour (or number) corresponds

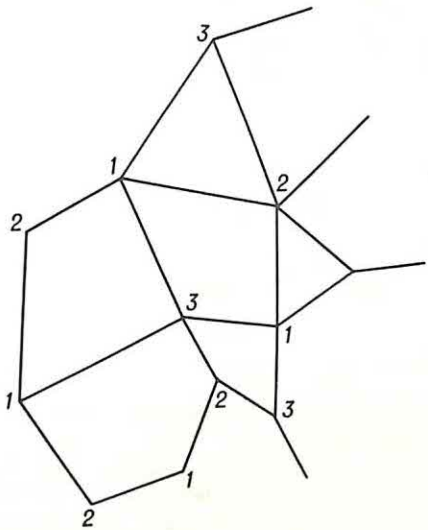

FIG. 41

to each vertex, such that no two "neighbouring" vertices (i. e. vertices joined by an edge) are coloured with the same colour (are numbered with the same number).

Let us now consider any (plane geographical) map. Select inside each country a certain point, its "capital". Then connect the

capitals by a railway network in such a way that whenever two countries are neighbours there is a railway line joining their capitals across the common border (Fig. 42). The map (planar graph) formed by the capitals of the countries and the railway lines is said to be *dual* to the original map (graph). Notice

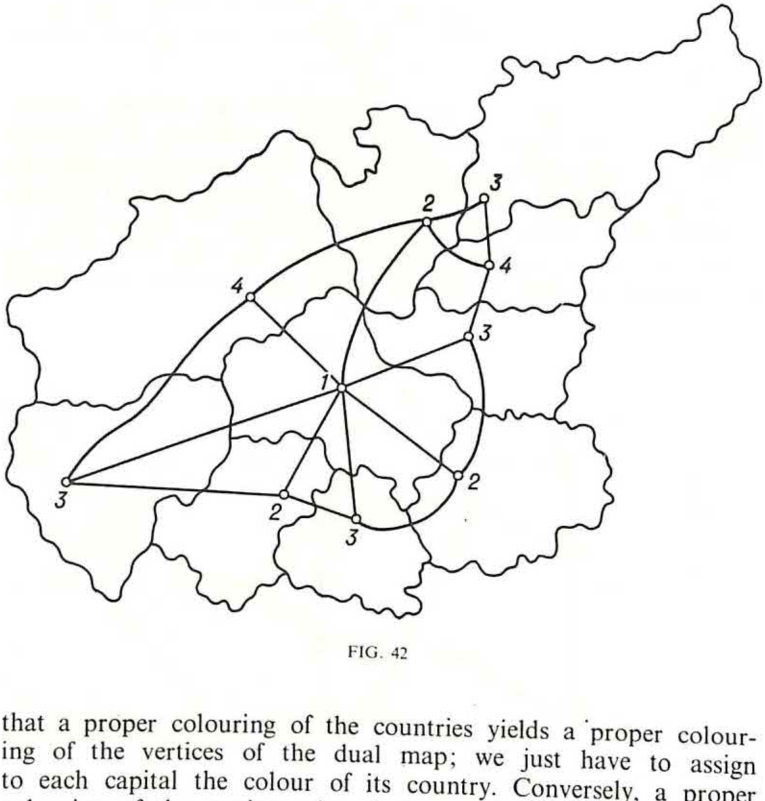

FIG. 42

that a proper colouring of the countries yields a proper colouring of the vertices of the dual map; we just have to assign to each capital the colour of its country. Conversely, a proper colouring of the vertices of a dual map is obviously equivalent to a proper colouring of the initial map. Thus, instead of speaking of a proper colouring of the countries of a map, one may speak of a proper colouring of the vertices of the dual map*. This

* The least number of colours required for a proper colouring of the vertices of an arbitrary graph (it is more convenient, of course, to speak of numbers with which we can properly number the vertices) is called the *chromatic number* of a graph.

purely technical detail (it is discussed, for instance, in [8]) has been used by all authors interested in the problem of map colouring.

PROBLEM 16. Let $\bar{K}$ be the dual map of K. Prove the converse, i. e. that K is dual to $\bar{K}$.

Hint. Compare with Fig. 42.

Let us now pass over to Kempe's proof of the five-colour theorem (Example 18). Kempe showed that in every normal map there is at least one country with two, three, four or five neighbours (Fig. 43). (In other words, there are no normal maps on a plane in which every country has six or more neighbours.) This may be expressed by saying that the set consisting of the "configurations": a country with two neighbours (a "bigonal" country), a country with three neighbours (a "triangular" country), a country with four neighbours (a "quadrangular" country) and a country with five neighbours (a "pentagonal" country), is "*unavoidable*", that is, *every normal map must contain at least one of these* four configurations. As a rule, attempts to prove the four-colour theorem reduce to considering these configurations.

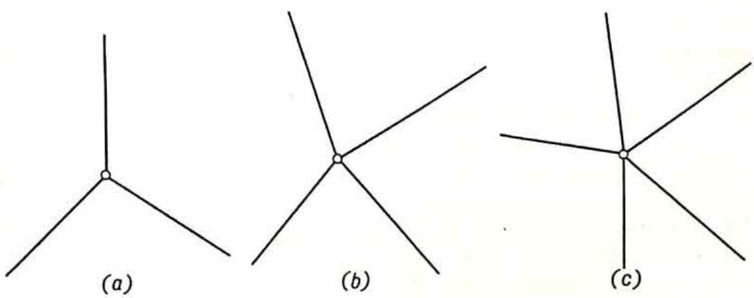

FIG. 43

For the five-colour theorem the unavoidable set of configurations found by Kempe turned out to be *reducible*, in the sense that each map containing one of the mentioned configurations and colourable with five colours can be reduced to a map containing a *smaller* number of countries, which can also be properly coloured with five colours. This enabled Kempe to use the method of mathematical induction (on the number n of countries) in his proof of the five-colour theorem. He conjectured that he had

found such an unavoidable, and at the same time, reducible set of configurations for the four-colour conjecture. In 1890, 11 years after Kempe published his proof, Heawood pointed out a flaw in Kempe's argument that no minimal five-chromatic map could contain a country with five neighbours. The error did not appear easy to repair.

Heinrich Heesch of the University of Hannover seems to have been the first mathematician after Kempe to publicly state that the four-colour conjecture could be proved by finding an unavoidable set of reducible configurations. He began his work on the conjecture in 1936, and made several major contributions to the existing theory. He was the first to seriously suggest that in the age of computers it is no longer important that the required unavoidable set of reducible configurations should be very small, since today the proof of reducibility can be carried out using a computer, which can try out all the possibilities much more quickly than man. An American mathematician Kenneth Appel was among those who assisted Heesch in his work on the four-colour conjecture with the aid of computers. But their co-activity ceased when in October 1971 it was rumoured that the four-colour problem had been already solved with the aid of a computer. It turned out that the news was somewhat premature. Appel, who had returned by that time to the United States, continued the same work in cooperation with Wolfgang Haken using an IBM 360 computer at the University of Illinois. He was assisted by a team of programmers headed by John Koch.

In June, 1976, these co-workers (Appel-Haken-Koch-IBM 360) completed their construction of an unavoidable set of reducible configurations, consisting of 1482 rather complex configurations. The existence of this set and the obvious fact that each map of not more than four countries, can be properly coloured with four colours establish, by the induction principle, the validity of the four-colour conjecture. Thus, today we may apparently consider the four-colour theorem as proved (see [1]). The word "apparently" in the last sentence underlines that as far as the completeness of the proof is concerned, we will have to "believe" the IBM 360 computer, since it is clearly impossible for us to check the whole reasoning "by hand" without a computer. By the way, in 1978 the American mathematician Daniel Cohen essentially completed the Appel-Haken argument, since the solution of the colouring problem suggested by him can be fully checked without using a computer. (Cohen calls it a "human solution".) (Of course, the computer played an important part

in finding the solution, however the solution found by the computer may be stated and checked without recourse to a computer. It is also interesting to note that the work of Appel and Haken required many thousands of hours of computing time, while Cohen's programme took only seconds.)

The solution of the four-colour problem was an outstanding event not only because one of the most famous mathematical hypothesis was proved, but also because it demonstrated the force of collaboration between men and computers.

Let us finally note that the results obtained by Ringel and Youngs, and Appel and Haken do not exhaust all aspects of the map colouring problem, a problem which has been thoroughly studied by many mathematicians over the past century from different points of view. Thus, for instance, there still remains unanswered the naturally arising question of the *least number of colours required for a proper colouring of any map on two globes* (i. e. on two planets). Each country may consist of two regions (pieces) situated on different planets. It is only natural to colour these regions with one and the same colour. If, as in the four-colour theorem, we consider that none of the regions situated on one planet is broken into (unconnected) parts, then we come across examples which show that there exist maps on two globes which cannot be properly coloured with fewer than 8 colours. It has only been proved so far that any such map is colourable with 12 colours (see, for instance [17]). (It goes without saying that the problem can be stated in a more complicated way, for instance, by increasing the number of globes or by varying their shape. For example: What is the least number of colours sufficient for a proper colouring of any map on two torus-shaped planets or a map on Saturn and on its ring?)

Sec. 3. Construction by Induction

The method of mathematical induction can be used to solve a geometric construction problem if the initial condition of the problem contains a certain positive integer n (for instance, in problems on constructing n-gons). Later on we consider a number of examples of this kind. The present section deals with self-intersecting polygons as well (Fig. 44). In other words, in most problems a polygon is understood as being *any* closed polygonal line $A_1A_2 \ldots A_n$.

EXAMPLE 21. Given: $2n+1$ points in a plane. Construct a $(2n+1)$-gon, for which the given points serve as the midpoints of the sides.

Solution. 1°. For $n = 1$ the problem is reduced to constructing a triangle from the given midpoints of its sides. This is easily solved (it is sufficient to draw through each of the three given points a straight line parallel to the straight line joining the other two points).

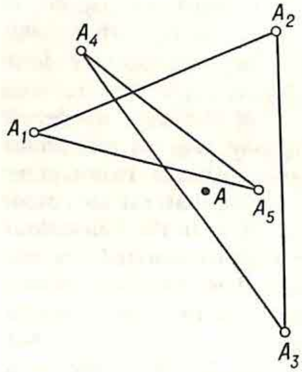

FIG. 44

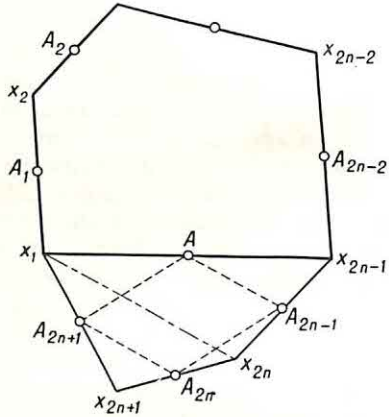

FIG. 45

2°. Let us assume that we can construct a $(2n-1)$-gon from the given midpoints of its sides, and let there be given $2n+1$ points $A_1, A_2, \ldots A_{2n+1}$ which are the midpoints of the sides of the required $(2n+1)$-gon $x_1x_2 \ldots x_{2n+1}$.

Consider the quadrilateral $x_1x_{2n-1}x_{2n}x_{2n+1}$ (Fig. 45). The points A_{2n-1}, A_{2n}, A_{2n+1} serve as the midpoints of its sides $x_{2n-1}x_{2n}$, $x_{2n}x_{2n+1}$, $x_{2n+1}x_1$. Let A be the midpoint of the side x_1x_{2n-1}. The quadrilateral $A_{2n-1}A_{2n}A_{2n+1}A$ is a parallelogram (to prove this it is sufficient to draw the straight line x_1x_{2n} and to consider the triangles $x_1x_{2n+1}x_{2n}$ and $x_1x_{2n-1}x_{2n}$ for which the line segments $A_{2n}A_{2n+1}$ and $A_{2n-1}A$, respectively, serve as middle lines). Since the points A_{2n-1}, A_{2n} and A_{2n+1} are known, the fourth vertex A of the parallelogram can readily be constructed. The points $A_1, A_2, \ldots, A_{2n-2}, A$ are the midpoints of the $(2n-1)$-gon $x_1x_2 \ldots x_{2n-1}$ which we can construct by assumption. To complete the solution construct the line segments x_1x_{2n+1} and $x_{2n-1}x_{2n}$

(the points x_1 and x_{2n-1} being already determined) which are bisected at the known points A_{2n+1} and A_{2n-1}.

In the case of a polygon having no self-intersections it is clear what is meant by the points exterior and interior relative to this polygon. In the general case this notion looses its meaning. For instance, we cannot say for sure whether the point A in Fig. 44 is situated inside or outside the polygon. Instead, let us introduce the following definition. Let there be given an arbitrary polygon $A_1A_2 \ldots A_n$, and let us establish for it a definite *direction of traversing* its vertices (say, in the order $A_1, A_2, \ldots, A_n$). Suppose that on one of the sides of the polygon, say A_1A_2, a triangle A_1BA_2 is constructed. If the direction in which the vertices of the triangle are traversed in the order A_1, A_2, B is opposite to the direction of traversing the vertices of the polygon (the former being clockwise, and the latter anticlockwise), then we shall say that the triangle is oriented *outside* the polygon. But if the two directions coincide, then the triangle is said to be oriented *inside* the polygon.

EXAMPLE 22. Given: n points in a plane. Construct an n-gon whose sides are the bases of isosceles triangles having their vertices at the n given points and vertex angles $\alpha_1, \alpha_2, \ldots, \alpha_n$*.

Solution. Let us assume that some of the angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ may even exceed $180°$, provided that for $\alpha < 180°$ the corresponding isosceles triangle is oriented outwards the polygon and for $\alpha > 180°$ it is oriented inside (the vertex angle in this case being equal to $360° - \alpha$).

1°. Let $n = 3$. Suppose that the problem is solved; x_1, x_2, x_3 are the vertices of the required triangle, and A_1, A_2, A_3 are the given vertices of the isosceles triangles constructed on its sides with the vertex angles α_1, α_2, α_3 (Fig. 46, *a*). By rotating the plane about the point A_1 through an angle α_1 (let us agree to the convention that all rotations are carried out anticlockwise) the vertex x_1 is moved to x_2, and by rotating about the point A_2 through the angle α_2, the vertex x_2 will move to x_3. The two rotations carried out in succession are equivalent to a single rotation through an angle $\alpha_1 + \alpha_2$ about a point A, which can be constructed from the points A_1 and A_2, and the angles α_1 and α_2 in the following way: on the line segment A_1A_2 construct the angles $\alpha_1/2$ and $\alpha_2/2$ at the points A_1 and A_2 respectively.

* The preceding example may be considered as a particular case of Example 22 for $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 180°$.

The point A where the second sides of these angles intersect will be the centre of the resultant rotation through the angle $\alpha_1 + \alpha_2$ (see [25], Chapter I, § 2). This resultant rotation brings the vertex x_1 to x_3. Consequently, the vertex x_3 is brought to x_1 by rotating the plane about the point A through the angle of $360° - (\alpha_1 + \alpha_2)$ and, hence, this point is the vertex of the isosceles triangle with the base x_1x_3 and the vertex angle $360 - (\alpha_1 + \alpha_2)$.

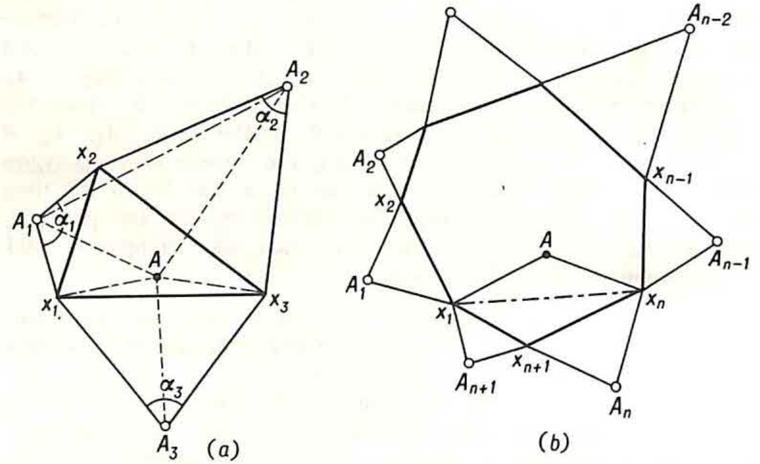

FIG. 46

Using the points A and A_3 (if they do not coincide – which can occur only if $\alpha_1 + \alpha_2 + \alpha_3 = 360° \cdot k$), we can construct the side x_1x_3. To this end, we construct on the line segment AA_3 angles of $\dfrac{360° - (\alpha_1 + \alpha_2)}{2}$ and $\dfrac{\alpha_3}{2}$ on both sides from the points A and A_3 respectively. The points at which the sides of the angles intersect will be the vertices x_1 and x_3 of the required triangle. It is now easy to construct its third vertex x_2. When $\alpha_1 + \alpha_2 + \alpha_3 = 360° \cdot k$ (i. e. when the point A coincides with A_3) the solution of the problem is indeterminate.

2°. Suppose that we know how to construct an n-gon from the vertices of the isosceles triangles constructed on its sides with the given vertex angles, and suppose it is required to construct an $(n+1)$-gon given the vertices $A_1, A_2, \dots A_n, A_{n+1}$ of the isosceles triangles constructed on its sides with vertex angles α_1, $\alpha_2, \dots, \alpha_n, \alpha_{n+1}$.

Let $x_1x_2 \ldots x_nx_{n+1}$ be the required $(n+1)$-gon (Fig. 46, *b*). Consider the triangle $x_1x_nx_{n+1}$. As in 1°, knowing the vertices A_n and A_{n+1} of the isosceles triangles $x_nA_nx_{n+1}$ and $x_{n+1}A_{n+1}x_1$ constructed on the sides x_nx_{n+1} and $x_{n+1}x_1$, we can find the vertex A of the isosceles triangle x_1Ax_n constructed on the diagonal x_1x_n and having the vertex angle equal to $360° - (\alpha_n + \alpha_{n+1})$. Thus, our problem is reduced to the problem of constructing an n-gon $x_1x_2 \ldots x_n$ from the vertices $A_1A_2 \ldots A_{n-1}A$ of the isosceles triangles constructed on its sides with known vertex angles $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, 360° - (\alpha_n + \alpha_{n+1})$. By the induction assumption, the n-gon $x_1x_2 \ldots x_n$ can be constructed, after which it is not difficult to constuct the required $(n+1)$-gon $x_1x_2 \ldots x_nx_{n+1}$.

For $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 360° \cdot k$ the solution of the problem is impossible or indeterminate (why?).

PROBLEM 17. Given: n points in a plane. Construct an n-gon for which the given points are the vertices of triangles constructed on its sides, each triangle having the ratio of the two sides not coinciding with the side of the given n-gon, and the vertex angle between these two sides given.

Hint. This problem can be solved analogously to the preceding problem (representing its particular case), but instead of rotating the plane about a given point A_1 through a known angle α_1, here we have to consider a similarity transformation consisting of a rotation through an angle α_1 and a homothetic transformation about the same centre A_1 and with the ratio of magnification equal to the ratio of the sides of the corresponding triangle (and analogously for the other given points). Two such transformations in succession are equivalent to a third transformation of the same kind (see, for instance [26], Chapter I, § 2). Consequently, using notation analogous to that of the previous example, from the vertices A_1 and A_2 of the triangles $x_1x_2A_1$ and $x_2x_3A_2$ we can find the vertex A of the triangle x_1x_3A constructed on the line segment x_1x_3 and having a known vertex angle and a known ratio of the two sides.

The side x_1x_3 of the triangle $x_1x_2x_3$ from the points A and A_3 can be constructed, for instance, in the following way. Two similarity transformations carried out in succession (with centres A and A_3) takes x_1 to itself (first x_1 is taken to x_3, and then x_3 to x_1). But the succession of these transformations is equivalent to a single similarity transformation with centre at some point B, which can be constructed. Since the point B is transformed to itself, it coincides with the required point x_1. If the sum of the vertex angles is a multiple 360°, and the product of the ratios of the sides is equal to unity, then the solution of the problem is impossible or indeterminate.

EXAMPLE 23. Given: a circle and n points in a plane. Inscribe in the circle an n-gon whose sides pass through the given points.

Solution. This problem is a difficult one. To solve it we have to use the method of mathematical induction in a quite unexpected way. The reason is that here we are unable to make use of induction on the number n of sides of the polygon. Instead, we have to consider the more general problem of constructing an n-gon, k neighbouring sides of which pass through k given points, the remaining $n-k$ sides being parallel to given straight lines (this problem reduces to the problem under consideration for $k = n$), and then *carry out induction on the number* k.

1°. For $k = 1$ we have the following problem: inscribe in a circle an n-gon, whose side A_1A_n passes through a given point P, and whose remaining $n-1$ sides $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ are parallel to the given straight lines $l_1, l_2, \ldots, l_{n-1}$.

Suppose the problem is solved and the required polygon is constructed (Fig. 47, a, b). Take an arbitrary point B_1 on the circle and construct an inscribed polygon $B_1B_2 \ldots B_n$, whose sides $B_1B_2, B_2B_3, \ldots, B_{n-1}B_n$ are parallel to the lines $l_1, l_2, \ldots, l_{n-1}$, respectively. Then the arcs $A_1B_1, A_2B_2, \ldots, A_nB_n$ will be equal

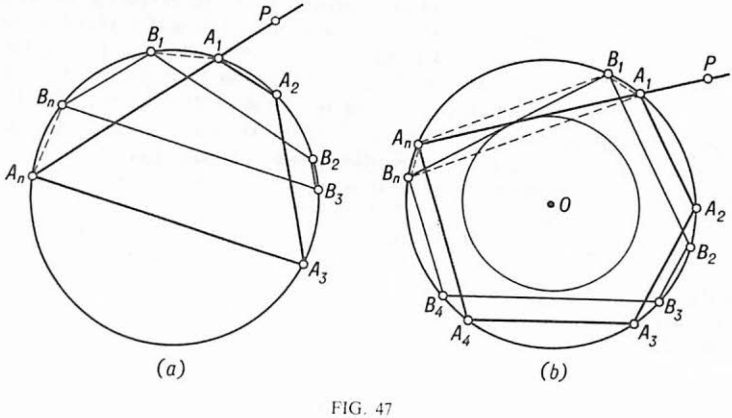

FIG. 47

to each other, the arcs A_1B_1 and A_2B_2, A_2B_2 and A_3B_3 and so on having opposite directions on the circle. Consequently, when n is even the arcs A_1B_1 and A_nB_n are oppositely directed, and the quadrilateral $A_1B_1B_nA_n$ is an isosceles trapezoid with the bases A_1A_n and B_1B_n (Fig. 47, a). Therefore, the side A_1A_n of the required polygon is parallel to the side B_1B_n of the n-gon $B_1B_2 \ldots B_n$. Consequently, in this case we have to draw a straight line parallel

to B_1B_n through the point P, and then it is quite easy to determine the remaining vertices of the n-gon $A_1A_2 \ldots A_n$ (do this!).

When n is odd the arcs A_1B_1 and A_nB_n have the same direction, and the quadrilateral $A_1B_1A_nB_n$ is an isosceles trapezoid with bases A_1B_n and B_1A_n (Fig. 47, b). Since the diagonals A_1A_n and B_1B_n of the trapezoid are equal to each other, in this case we have to draw through the point P a straight line on which the given circle cuts off a chord A_1A_n equal to the known chord B_1B_n, i.e. a straight line tangent to the circle concentric with the given one and touching B_1B_n (do this!).

2°. Suppose that we can already solve the problem of constructing an inscribed n-gon, k successive sides of which pass through k given points, the remaining $n-k$ sides being parallel to the given lines, and let it be required to inscribe in a circle an n-gon in which $k+1$ neighbouring sides $A_1A_2, A_2A_3, \ldots, A_{k+1}A_{k+2}$ pass through $k+1$ given points $P_1, P_2, \ldots, P_{k+1}$, the remaining $n-k-1$ sides being parallel to the given lines.

Suppose that the problem is solved, and the required n-gon is constructed (Fig. 48). Consider the sides A_1A_2 and A_2A_3 of this polygon. Draw through the vertex A_1 a straight line A_1A_2' parallel to P_1P_2. Denote the point of intersection of this line with the circle by A_2' and the point of intersection of the line $A_2'A_3$ with P_1P_2 by P_2'. The triangles $P_1A_2P_2$ and $P_2'P_2A_3$ are similar, since $\angle A_2P_1P_2 = \angle A_2A_1A_2' = \angle A_2A_3P_2'$ and $\angle A_2P_2P_1 = \angle P_2'P_2A_3$. Consequently, $\dfrac{P_1P_2}{A_3P_2} = \dfrac{A_2P_2}{P_2'P_2}$, from which we have

$$P_2'P_2 = \frac{A_3P_2 \cdot A_2P_2}{P_1P_2}.$$

Since the product $A_3P_2 \cdot A_2P_2$ depends only on the given point P_2 and on the circle (but not on the choice of the points A_2 and A_3!), it can be determined. Therefore the length of the segment $P_2'P_2$ can be found and hence the point P_2' can be constructed. Thus, we know k points $P_2', P_3, \ldots, P_{k+1}$ through which there pass k neighbouring sides $A_2'A_3, A_3A_4, \ldots, A_{k+1}A_{k+2}$ of the n-gon $A_1A_2'A_3 \ldots A_n$, the remaining $n-k$ sides $A_{k+2}A_{k+3}, \ldots, A_nA_1$, A_1A_2' being parallel to the known lines. By virtue of the induction assumption, we can construct the n-gon $A_1A_2'A_3 \ldots A_n$, and then, it is not difficult to construct the required n-gon $A_1A_2 \ldots A_n$.

PROBLEM 15. In a given circle inscribe an n-gon whose k sides (not necessarily neighbouring!) pass through k given points, the remaining $n-k$ sides being parallel to the given lines.

Hint. Let the side A_1A_2 of the required polygon pass through the point P, and the side A_2A_3 be parallel to the straight line l (Fig. 49). Let P' denote the point symmetrical to the point P about the circle diameter perpendicular to l, and A'_2 the point of intersection

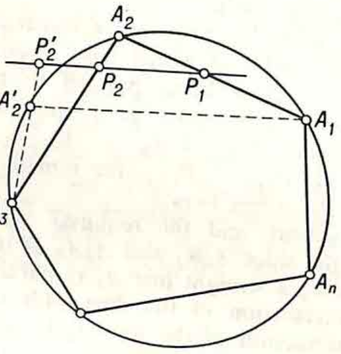

FIG. 48

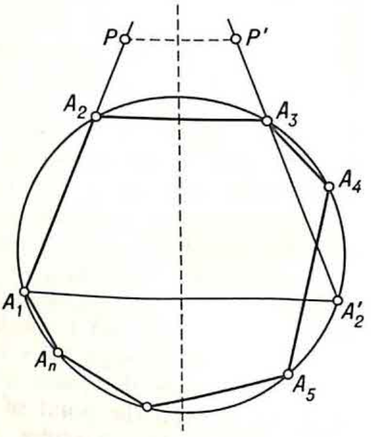

FIG. 49

of the line $P'A_3$ with the circle. In the n-gon $A_1A'_2A_3 \ldots A_n$ the side $A_1A'_2$ is parallel to the given line l, and the side A'_2A_3 passes through the known point P'. Performing this construction the required number of times, we reduce this problem to that of constructing an n-gon in which k *neighbouring* sides pass through the known points, the remaining $n-k$ sides being parallel to the given straight lines.

EXAMPLE 24. Given: two parallel straight lines l and l_1. Using only a ruler, divide the segment AB of the line l into n equal parts (Fig. 50).

Solution. 1°. Let $n=2$. Join an arbitrary point S in plane to the points A and B (Fig. 50, *a*), and denote by C and D respectively the points of intersection of the lines AS and BS with the line l_1. Denote the point of intersection of the straight lines AD and BC by T_2 and the point of intersection of the straight lines ST_2 and l by P_2. We shall prove that P_2 is the required point, i.e. $AP_2 = \dfrac{1}{2}AB$.

Let Q_2 denote the point of intersection of the lines ST_2 and l_1. It is easy to see that

$$\triangle\, T_2P_2B \sim \triangle\, T_2Q_2C,\quad \triangle\, ABT_2 \sim \triangle\, DCT_2,$$
$$\triangle\, SAP_2 \sim \triangle\, SCQ_2 \text{ and } \triangle\, SAB \sim \triangle\, SCD,$$

whence

$$\frac{P_2B}{Q_2C} = \frac{T_2B}{T_2C} = \frac{AB}{CD} \text{ and } \frac{P_2A}{Q_2C} = \frac{SA}{SC} = \frac{AB}{CD}.$$

Consequently,

$$\frac{P_2B}{Q_2C} = \frac{P_2A}{Q_2C},$$

and therefore $P_2A = P_2B$ and $AP_2 = \frac{1}{2}AB$.

2°. Suppose that we can already, using only a ruler, construct a point P_n of the line segment AB such that $AP_n = \frac{1}{n}AB$. Take an arbitrary point S not on the lines l or l_1, and denote by T_n and Q_n the points of intersection of the straight line SP_n

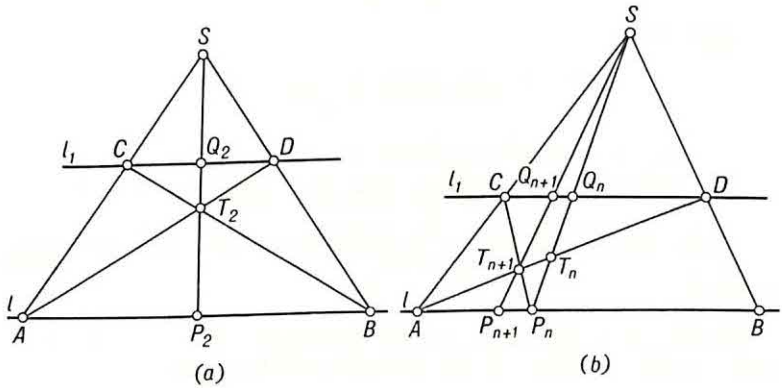

FIG. 50

with AD and l_1, respectively (Fig. 50, b). Join the point T_{n+1} of intersection of AD and CP_n to S, and denote the points of intersection of ST_{n+1} with the lines l_1 and l by Q_{n+1} and P_{n+1}.

We prove that P_{n+1} is the required point, i.e. $AP_{n+1} = \frac{1}{n+1}AB$.

Indeed, since the triangles $CQ_{n+1}T_{n+1}$ and $P_nP_{n+1}T_{n+1}$, $CT_{n+1}D$ and $P_nT_{n+1}A$ are similar, we have:

$$\frac{P_{n+1}P_n}{CQ_{n+1}} = \frac{P_nT_{n+1}}{CT_{n+1}} = \frac{AP_n}{CD}; \tag{11}$$

from similarity of the triangles SAP_{n+1} and SCQ_{n+1}, SAB and SCD we have

$$\frac{AP_{n+1}}{CQ_{n+1}} = \frac{SA}{SC} = \frac{AB}{CD}. \tag{12}$$

From the equalities (11) and (12) we have

$$\frac{P_{n+1}P_n}{AP_{n+1}} = \frac{AP_n}{AB}$$

or, since

$$P_{n+1}P_n = AP_n - AP_{n+1} \text{ and } AP_n = \frac{1}{n}AB,$$

$$\frac{\frac{1}{n}AB - AP_{n+1}}{AP_{n+1}} = \frac{\frac{1}{n}AB}{AB}, \quad \frac{1}{n}AB - AP_{n+1} = \frac{1}{n}AP_{n+1},$$

whence, finally,

$$AP_{n+1} = \frac{1}{n+1}AB.$$

To find the successive points (of division) P'_{n+1}, P''_{n+1}, ..., it is sufficient to construct the line segments $P_{n+1}P'_{n+1} = \frac{1}{n}P_{n+1}B$, $P'_{n+1}P''_{n+1} = \frac{1}{n-1}P'B$ and so on, using the same method.

PROBLEM 19. Using a pair of compasses set to a distance a and a ruler, construct a line segment equal to a/n.

Hint. On a circle of radius a mark the points A_1, A_2, A_3, A_4, A_5, A_6 which are the vertices of a regular hexagon. Suppose we already know the point B_n on the radius OA_n such that $OB_n = \frac{1}{n}OA_n = \frac{a}{n}$ (we put here $A_{6m+k} = A_k$ for any m and $k = 1, 2, 3, 4, 5, 6$; $B_1 = A_1$). Let us denote by B_{n+1} the point of intersection of the lines OA_{n+1} and B_nA_{n+2}; then $OB_{n+1} = \frac{a}{n+1}$.

Sec. 4. Finding Loci by Induction

Let us consider a number of problems on finding loci with the aid of the method of mathematical induction.

EXAMPLE 25. Line segments $B_1C_1, B_2C_2, \ldots, B_nC_n$ are laid off on the sides of a convex n-gon $A_1A_2 \ldots A_n$. Find the locus of the interior points M of this polygon for which the sum of the areas of the triangles $MB_1C_1, MB_2C_2, \ldots, MB_nC_n$ is constant (and is equal to the sum $S_{\triangle M_0B_1C_1} + S_{\triangle M_0B_2C_2} + \ldots + S_{\triangle M_0B_nC_n}$ where M_0 is a definite point inside the polygon).

Solution. 1°. Let $n = 3$ (Fig. 51, *a*). On the sides A_3A_2 and A_3A_1 of the triangle $A_1A_2A_3$ lay off line segments $A_3P = B_2C_2$ and $A_3Q = B_3C_3$. Then*

$$S_{\triangle M_0B_2C_2} + S_{\triangle M_0B_3C_3} = S_{\triangle M_0PA_3} + S_{\triangle M_0QA_3} = S_{\triangle PQA_3} + S_{\triangle M_0PQ}$$

and hence

$$S_{\triangle M_0B_1C_1} + S_{\triangle M_0B_2C_2} + S_{\triangle M_0B_3C_3} = S_{\triangle PQA_3} + (S_{\triangle M_0B_1C_1} + S_{\triangle M_0PQ}).$$

Analogously,

$$S_{\triangle MB_1C_1} + S_{\triangle MB_2C_2} + S_{\triangle MB_3C_3} = S_{\triangle PQA_3} + (S_{\triangle MB_1C_1} + S_{\triangle MPQ}).$$

We see that the required locus is defined by the following condition

$$S_{\triangle MB_1C_1} + S_{\triangle MPQ} = S_{\triangle M_0B_1C_1} + S_{\triangle M_0PQ}.$$

Let now N be the point of intersection of the straight lines A_1A_2 and PQ (if these lines are parallel, then the required locus will obviously be a segment of a straight line parallel to them). Let us lay off on the sides of the angle A_2NP line segments $NR = PQ$ and $NS = B_1C_1$. Then

$$S_{\triangle M_0B_1C_1} + S_{\triangle M_0PQ} = S_{\triangle M_0NS} + S_{\triangle M_0NR} = S_{\triangle NRS} + S_{\triangle M_0RS}$$

and analogously

$$S_{\triangle MB_1C_1} + S_{\triangle MPQ} = S_{\triangle NRS} + S_{\triangle MRS}$$

Consequently, the required locus consists of those points M lying inside the triangle for which $S_{\triangle MRS} = S_{\triangle M_0RS}$, i. e. it represents

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* Here we suppose that the point M_0 lies inside the quadrilateral A_1A_2PQ; if this is not so the reasoning does not change much.

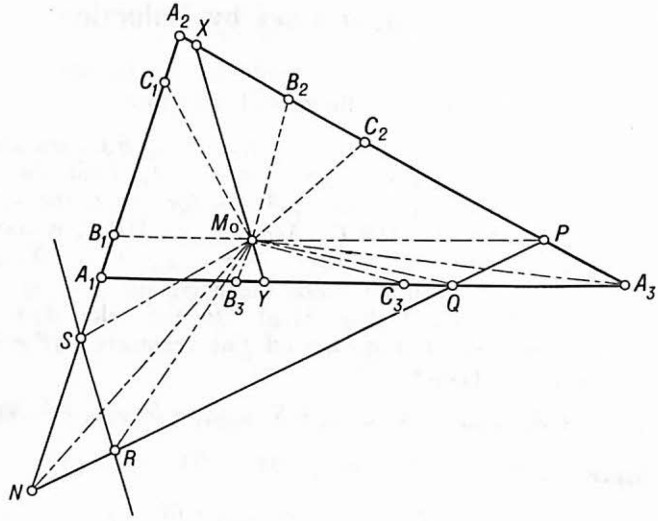

FIG. 51a

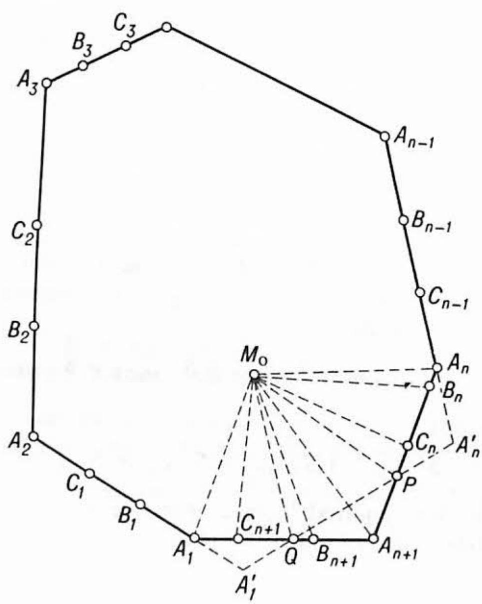

FIG: 51b

a segment XY of the straight line passing through the point M_0 (and parallel to the straight line RS*).

2°. Suppose we already know that the required locus for the n-gon is a segment of a straight line (passing, of course, through the point M_0). Let us now consider the $(n+1)$-gon $A_1A_2 \ldots A_nA_{n+1}$; let $B_1C_1, B_2C_2, \ldots, B_nC_n, B_{n+1}C_{n+1}$ be the given line segments laid off on its sides and M_0 a point inside $(n+1)$-gon (Fig. 51, *b*). On the sides of the angle $A_1A_{n+1}A_n$ lay off the segments $A_{n+1}P = B_nC_n$ and $A_{n+1}Q = B_{n+1}C_{n+1}$ (from the vertex A_{n+1}). Then

$$S_{\triangle MB_nC_n} + S_{\triangle MB_{n+1}C_{n+1}} = S_{\triangle MA_{n+1}P} + S_{\triangle MA_{n+1}Q} = \\ = S_{\triangle A_{n+1}PQ} + S_{\triangle MPQ}.$$

Hence, for the points M of the required locus

$$S_{\triangle MB_1C_1} + S_{\triangle MB_2C_2} + \ldots + S_{\triangle MB_{n-1}C_{n-1}} + S_{\triangle MPQ} = \\ = S_{\triangle M_0B_1C_1} + S_{\triangle M_0B_2C_2} + \ldots + S_{\triangle M_0B_{n-1}C_{n-1}} + S_{\triangle M_0PQ}.$$

By virtue of the induction assumption, the required locus turns out to be a segment of a straight line passing through the point M_0.

The solution of this problem gives us a hint on the method of constructing this locus.

PROBLEM 20. Given: n straight lines $l_1, l_2, \ldots, l_n$ with a line segment marked off on each of them ($B_1C_1, B_2C_2, \ldots, B_nC_n$, respectively) and a point M_0. Find the locus of points M for which the algebraic sum of the areas of the triangles MB_1C_1, $MB_2C_2, \ldots, MB_nC_n$, is equal to the corresponding sum for the point M_0, where the area of the triangle MB_iC_i ($i = 1, 2, \ldots, n$) is taken with a plus sign if the point M lies on the same side of the straight line l_i as the point M_0, and with a minus sign if otherwise.

Hint. The required locus is a straight line, the proof is similar to the solution of Example 25.

PROBLEM 21 (*Newton's Theorem*). Prove that the midpoints of the diagonals of a quadrilateral circumscribed about a circle

* We could begin the induction with the case $n = 2$, when the "n-gon" represents an angle having only two sides.

lie on one straight line passing through the centre of the circle (Fig. 52).

Hint. Using the notation of Fig. 52 we have:

$$S_{\triangle BCE} + S_{\triangle ADE} = S_{\triangle BCF} + S_{\triangle ADF} = S_{\triangle BCO} + S_{\triangle ADO} = \frac{1}{2} S,$$

where S is the area of the quadrilateral. Hence, by virtue of the result of Example 25 (or Problem 20) it follows that the points E, F and O lie on the same straight line.

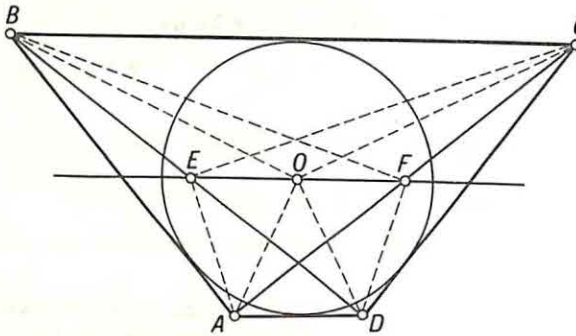

FIG. 52

PROBLEM 22 (*Gauss' Theorem*). Prove that the straight line joining the midpoints of the diagonals of a convex quadrilateral (which is neither a parallelogram, nor a trapezoid) bisects the line segment connecting the points of intersection of the opposite sides (Fig. 53).

Hint. Using the notation of Fig. 53 (where P is the midpoint of the line segment EF), we have:

$$S_{\triangle ABM} + S_{\triangle CDM} = S_{\triangle ABN} + S_{\triangle CDN} = S_{\triangle ABP} - S_{\triangle CDP} = \frac{1}{2} S,$$

where S is the area of the quadrilateral. Hence, by virtue of the result of Problem 20 it follows that the points M, N and P lie on one straight line.

EXAMPLE 26. Given: n points $A_1, A_2, \ldots, A_n$ and n (positive or negative!) numbers $a_1, a_2, \ldots, a_n$. Find the locus of points M for which the sum

$$a_1 \cdot MA_1^2 + a_2 \cdot MA_2^2 + \ldots + a_n \cdot MA_n^2$$

is constant.

Solution. 1°. Let $n = 2$. To be definite let us first assume that the numbers a_1 and a_2 are both positive.

Let us take the point O on the line segment A_1A_2 which divides it in the ratio $a_2 : a_1$, so that $OA_1 = \frac{a_2}{a_1 + a_2} A_1A_2$ and $OA_2 = \frac{a_1}{a_1 + a_2} A_1A_2$. Let M be an arbitrary point in a plane and H the foot of the perpendicular dropped from the point M onto

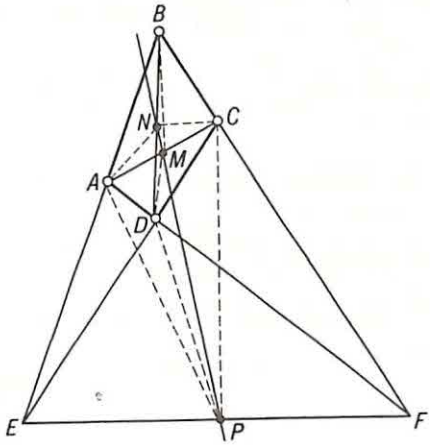

FIG. 53

the straight line A_1A_2 (Fig. 54). We then have

$$MA_1^2 = MO^2 + A_1O^2 \pm 2A_1O \cdot HO,$$
$$MA_2^2 = MO^2 + A_2O^2 \pm 2A_2O \cdot HO.$$

Multiplying the first of the equalities by A_2O, the second by A_1O and adding them termwise, we get:

$$MA_1^2 \cdot A_2O + MA_2^2 \cdot A_1O =$$
$$= MO^2 (A_2O + A_1O) + A_1O^2 \cdot A_2O + A_2O^2 \cdot A_1O =$$
$$= MO^2 \cdot A_1A_2 + A_1O \cdot A_2O \cdot A_1A_2.$$

Substituting now A_1O and A_2O by their values, we obtain:

$$MA_1^2 \frac{a_1 \cdot A_1A_2}{a_1 + a_2} + MA_2^2 \frac{a_2 \cdot A_1A_2}{a_1 + a_2} = MO^2 \cdot A_1A_2 +$$
$$+ \frac{a_1}{a_1 + a_2} \frac{a_2}{a_1 + a_2} A_1A_2^3,$$

or

$$a_1 MA_1^2 + a_2 MA_2^2 = (a_1 + a_2) MO^2 + \frac{a_1 a_2}{a_1 + a_2} A_1 A_2^2.$$

Consequently, if

$$a_1 MA_1^2 + a_2 MA_2^2 = R^2,$$

then

$$MO^2 = \frac{R^2}{a_1 + a_2} - \frac{a_1 a_2}{(a_1 + a_2)^2} A_1 A_2^2 = \text{const.}$$

Hence it follows that if $\frac{R^2}{a_1 + a_2} - \frac{a_1 a_2}{(a_1 + a_2)^2} A_1 A_2^2 > 0$, then the required locus is a circle of radius $\sqrt{\frac{R^2}{a_1 + a_2} - \frac{a_1 a_2}{(a_1 + a_2)^2} A_1 A_2^2}$ with the point O as centre; if $\frac{R^2}{a_1 + a_2} - \frac{a_1 a_2}{(a_1 + a_2)^2} A_1 A_2^2 = 0$, then the required locus consists of the single point O; finally, if $\frac{R^2}{a_1 + a_2} - \frac{a_1 a_2}{(a_1 + a_2)^2} A_1 A_2^2 < 0$, then the locus consists of no points at all.

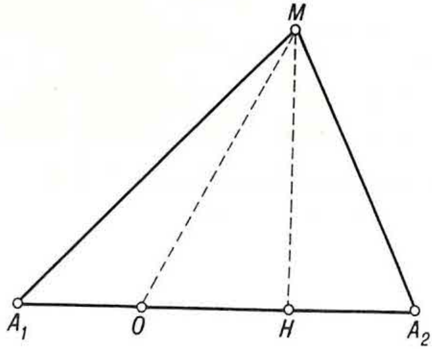

FIG. 54

The case when a_1 and a_2 are both negative obviously reduces to the preceding one. If $a_1 > 0$, $a_2 < 0$ and $a_1 + a_2 \neq 0$ (for instance, $a_1 + a_2 > 0$), then point O should be chosen on the extension of the line segment $A_1 A_2$ to the right of the point A_2

so that $A_2O = \left| \frac{a_1}{a_1 + a_2} \right|$ and $A_1O = \left| \frac{a_2}{a_1 + a_2} \right|$; the subsequent reasoning will be similar to that above. Finally, if $a_1 + a_2 = 0$, then $a_1 = -a_2$ and our problem is reduced to the following: find the locus of points M for which the difference between the squared distances from two given points A_1 and A_2 is constant. Let H be the foot of the perpendicular dropped from the point M on the straight line A_1A_2 (Fig. 54); then $MA_1^2 = MH^2 + A_1H^2$, $MA_2^2 = MH^2 + A_2H^2$ and, consequently, $MA_1^2 - MA_2^2 = A_1H^2 - A_2H^2$. If $MA_1^2 - MA_2^2 = R^2$, then $A_1H - A_2H = \frac{R^2}{A_1A_2}$, which completely defines point H. Hence it follows that in this case the required locus is a straight line passing through the point H and perpendicular to A_1A_2.

2°. Suppose that we have already proved that with n given points the corresponding locus is represented by a circle if $a_1 + a_2 + \ldots + a_n \neq 0$ or by a straight line if $a_1 + a_2 + \ldots + a_n = 0$. Let us now consider $n + 1$ points $A_1, A_2, \ldots, A_{n+1}$ and $n + 1$ numbers $a_1, a_2, \ldots, a_{n+1}$. Assume that $a_n + a_{n+1} \neq 0$ (if $a_n + a_{n+1} = 0$ we would replace this pair of numbers by the pair a_{n-1} and a_{n+1}, or by a_{n-1} and a_n. If simultaneously $a_n + a_{n+1} = 0$, $a_{n-1} + a_{n+1} = 0$ and $a_{n-1} + a_n = 0$, then $a_{n-1} = a_n = a_{n+1} = 0$, and we can directly make use of the induction assumption, since the problem then reduces to the case of $n - 2$ points $A_1, A_2, \ldots, A_{n-2}$ and $n - 2$ numbers $a_1, a_2, \ldots, a_{n-2}$).

As in 1°, let us show that on the line segment A_nA_{n+1} we can find a point O such that for any point M of the plane

$$a_nMA_n^2 + a_{n+1}MA_{n+1}^2 = (a_n + a_{n+1})\,MO^2 + \frac{a_na_{n+1}}{a_n + a_{n+1}}\,A_nA_{n+1}^2.$$

Hence, our problem is reduced to finding the locus of points M for which the sum

$$a_1MA_1^2 + a_2MA_2^2 + \ldots + a_{n-1}MA_{n-1}^2 + (a_n + a_{n+1})\,MO^2.$$

By the induction assumption, this locus will be a circle for $a_1 + a_2 + \ldots + a_n + a_{n+1} \neq 0$ and a straight line for $a_1 + a_2 + \ldots + a_n + a_{n+1} = 0$.

PROBLEM 23. Find the locus of points the sum of whose squared distances from n given points is constant.

Hint. It is sufficient to put $a_1 = a_2 = \ldots = a_n = 1$ in the initial condition of Example 26.

PROBLEM 24. Find the point for which the sum of its squared distances from n given points is minimal.

Hint. Consider the centre of the circle which is the required locus in Problem 23.

PROBLEM 25. Find the locus of points for which the ratio of the distances from it to two given points is constant.

Hint. If M is a point belonging to the required locus, then $\frac{AM}{BM} = c$ and, hence, $AM^2 - c \cdot BM^2 = 0$; therefore this problem is reduced to Example 26.

PROBLEM 26. Given: an n-gon $A_1A_2 \ldots A_n$. Find the locus of points M such that the polygon whose vertices are the projections of the point M onto the sides of the given polygon has a given area S.

Hint. It is easy to show that the area of the triangle whose vertices are the projections of the point M onto the sides of the triangle $A_1A_2A_3$ is equal to $\frac{1}{4}\left|1 - \frac{d^2}{R^2}\right| S_{\triangle A_1A_2A_3}$, where R is the radius of the circle $\sum$ circumscribed about the triangle $A_1A_2A_3$, and d the distance between the point M and the centre of the circle $\sum$. Hence, it follows that for $n = 3$ the required locus is a circle which is concentric with $\sum$ (or a pair of such circles). Then, by induction on the number of sides of the polygon we can show that for any n the required locus is, generally speaking, a circle (or a pair of concentric circles) (see the solution of Problem 90 from [21]).

Sec. 5. Definition by Induction

Interesting applications of the method of mathematical induction occur in problems containing notions whose definitions use the passage "from n to $n + 1$". Problems of this kind are considered in this section.

EXAMPLE 27. *Determination of the median and the centre of gravity of an n-gon.*

1°. We shall call the midpoint of a line segment its *centroid* (or "*centre of gravity*") (Fig. 55, *a*).

The *medians of the triangle* $A_1A_2A_3$ may then be defined as the line segments joining the vertices of the triangle to the centroids of the opposite sides (Fig. 55, *b*). As is known, the

medians of a triangle intersect in one point and are divided by this point in the ratio $2:1$ as measured from the vertex. The point at which the medians of a triangle intersect is called the *centroid of the triangle*.

Let us now agree to define the *medians of the quadrilateral* $A_1A_2A_3A_4$ as the line segments connecting each of its vertices A_1, A_2, A_3, A_4 with the centroids O_1, O_2, O_3, O_4 of the triangles

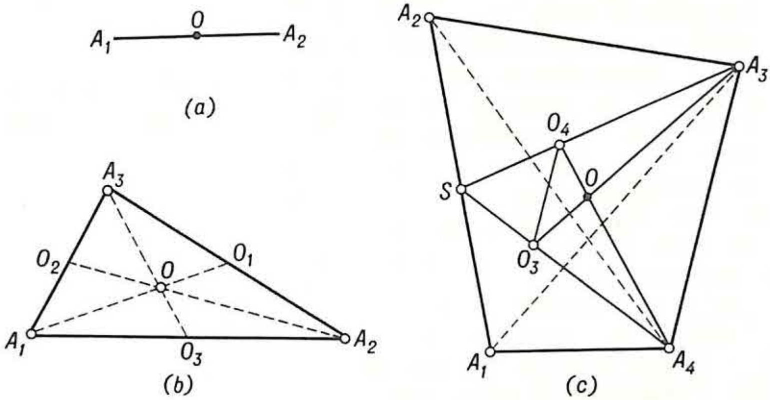

FIG. 55

formed by the remaining three vertices (Fig. 55, *c*). Let us prove that the *medians of a quadrilateral intersect in a single point and are divided by this point in the ratio* $3:1$, *as measured from their respective vertices*. In fact, let S denote the centroid (the midpoint) of the side A_1A_2, and O_4 and O_3 the centroids of the triangles $A_1A_2A_3$ and $A_1A_2A_4$, respectively; let O be the point of intersection of the medians A_3O_3 and A_4O_4 of the quadrilateral. Since SA_3 and SA_4 are the medians of the triangles $A_1A_2A_3$ and $A_1A_2A_4$, we may write

$$\frac{SA_3}{SO_4}=\frac{3}{1} \quad \text{and} \quad \frac{SA_4}{SO_3}=\frac{3}{1},$$

and, consequently,

$$\frac{SA_3}{SO_4}=\frac{SA_4}{SO_3}.$$

Hence, it follows that $O_3O_4 \parallel A_3A_4$ and $\dfrac{A_3A_4}{O_3O_4}=\dfrac{SA_3}{SO_4}=\dfrac{3}{1}$.

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Then, from similarity of the triangles OO_3O_4 and OA_3A_4 we have

$$\frac{OA_4}{OO_4} = \frac{OA_3}{OO_3} = \frac{A_3A_4}{O_3O_4} = \frac{3}{1}.$$

Thus, any two neighbouring medians of a quadrilateral are divided by the point of intersection in the ratio $3:1$. Hence it follows that all the four medians of a quadrilateral pass through the single point O which divides all of them in the ratio $3:1$. The point O where the medians of a quadrilateral intersect is called the *centroid of the quadrilateral*.

2°. Suppose that for all $k < n$ we have already defined the medians of the k-gon as the line segments joining the vertices of the k-gon to the centroids of the $(k-1)$-gons formed by the remaining $k-1$ vertices, and that for all $k < n$ we have defined the centroid of a k-gon as the point of intersection of its medians. We shall also assume as proved that for $k < n$ the medians of the k-gon are divided by the point of intersection (i. e. by the centroid of the k-gon) in the ratio $(k-1):1$ (taken from the vertex).

Let us now define the medians of an n-gon as the line segments connecting the vertices of the n-gon with the centroids of the $(n-1)$-gons formed by the remaining $n-1$ vertices. Let us prove that *all medians of the n-gon $A_1A_2 \ldots A_n$ intersect in a single point and are divided by this point in the ratio* $(n-1):1$ along the median. In fact, let S be the centroid of the $(n-2)$-gon $A_1A_2 \ldots A_{n-2}$; then the straight lines SA_{n-1} and SA_n will be the medians of the $(n-1)$-gons $A_1A_2 \ldots A_{n-1}$ and $A_1A_2 \ldots A_{n-2}A_n$ (Fig. 56). If O_n and O_{n-1} are the centroids of these $(n-1)$-gons, then, by the induction assumption,

$$\frac{SA_{n-1}}{SO_n} = \frac{SA_n}{SO_{n-1}} = \frac{n-1}{1}.$$

Hence, $O_{n-1}O_n \parallel A_nA_{n-1}$ and $\dfrac{A_{n-1}A_n}{O_{n-1}O_n} = \dfrac{n-1}{1}$. Let us denote the point of intersection of the medians $O_{n-1}A_{n-1}$ and O_nA_n of the n-gon $A_1A_2 \ldots A_n$ by O. From the similarity of the triangles $OO_{n-1}O_n$ and $OA_{n-1}A_n$ it follows that

$$\frac{OA_{n-1}}{OO_{n-1}} = \frac{OA_n}{OO_n} = \frac{A_{n-1}A_n}{O_{n-1}O_n} = \frac{n-1}{1}.$$

Thus, any two neighbouring medians of an n-gon are divided by the point of their intersection in the ratio $(n-1):1$. Hence, it follows that all the medians of an n-gon intersect in one point (and are divided by this point in the ratio $(n-1):1$).

We may now define the *centroid* of an n-gon as the point of intersection of its medians, and then the *medians* of an $(n+1)$-gon as the line segments connecting each of the vertices of an $(n+1)$-gon with the centroids of the n-gons formed by the n remaining vertices. The method of mathematical induction enables us to assert that our definitions of the medians and the centroid of an n-gon are meaningful for *any* n.

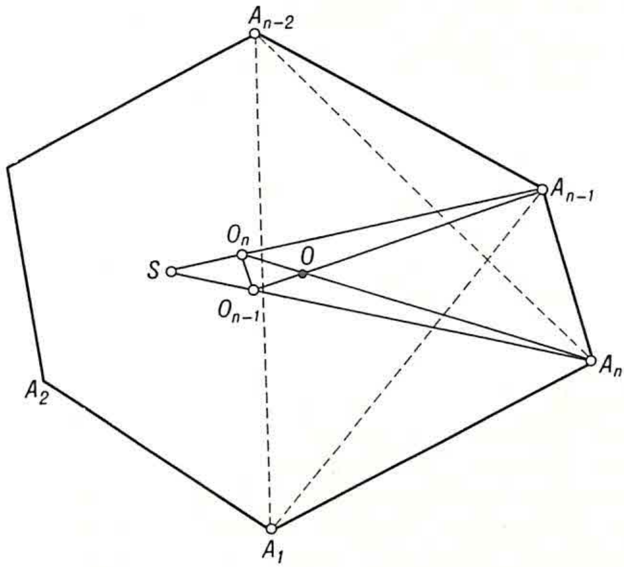


FIG. 56

PROBLEM 27. In the n-gon $A_1A_2 \ldots A_n$ let us denote the centroid of the $(n-1)$-gon $A_2A_3 \ldots A_n$ by O_1, the centroid of the $(n-1)$-gon $A_1A_3 \ldots A_n$ by O_2, and so on, till we have defined the centroid of the $(n-1)$-gon $A_1A_2 \ldots A_{n-1}$ by O_n. Prove that the n-gon $O_1O_2 \ldots O_n$ is similar to the given n-gon $A_1A_2 \ldots A_n$.

Hint. As is proved in Example 27, $O_1O_2 \parallel A_1A_2$ and $\dfrac{O_1O_2}{A_1A_2} = \dfrac{1}{n-1}$.

Analogously, $O_2O_3 \parallel A_2A_3$ and $\dfrac{O_2O_3}{A_2A_3} = \dfrac{1}{n-1}$, and so on.

We shall define a *median of the k-th order* of an n-gon $(k < n)$ as a line segment joining the centroid of the k-gon formed by any k vertices of the n-gon to the centroid of the $(n-k)$-gon

formed by the remaining $n-k$ vertices. Hence, a median of the k-th order is simultaneously also a median of the $(n-k)$-th order. The medians of the n-gon defined in Example 27 could be called *medians of the first order*.

PROBLEM 28. Prove that in an n-gon all medians of the k-th order intersect in one point and are divided by this point in the ratio $(n-k):k$.

Hint. Let S_1 and S_2 be respectively the centroids of the $(k-1)$-gon $A_2A_3 \ldots A_k$ and the $(n-k-1)$-gon $A_{k+2}A_{k+3} \ldots A_n$, O_1 and O_2 the centroids of the k-gons $A_1A_2 \ldots A_k$ and $A_2A_3 \ldots A_{k+1}$, and O_3 and O_4 the centroids of the $(n-k)$-gons $A_{k+1} \ldots A_n$ and $A_{k+2} \ldots A_nA_1$ (Fig. 57).

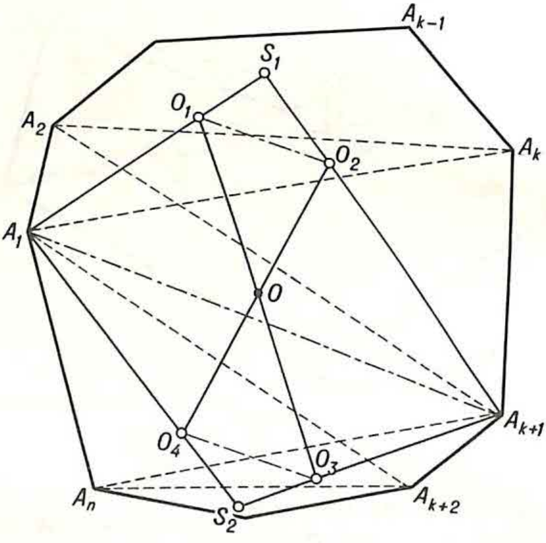


FIG. 57

Then $\dfrac{O_1S_1}{O_1A_1} = \dfrac{O_2S_1}{O_2A_{k+1}} = \dfrac{1}{k-1}$ and $O_1O_2 \parallel A_1A_{k+1}$; $\dfrac{O_3S_2}{O_3A_{k+1}} = \dfrac{O_4S_2}{O_4A_1} =$ $= \dfrac{1}{n-k-1}$ and $O_3O_4 \parallel A_1A_{k+1}$. If now O is the point of intersection of the *k-th* order medians O_2O_4 and O_1O_3, then from the similarity of

the triangles OO_1O_2 and OO_3O_4 we have

$$\frac{OO_1}{OO_3} = \frac{OO_2}{OO_4} = \frac{O_1O_2}{O_3O_4} = \frac{\frac{1}{k} A_1A_{k+1}}{\frac{1}{n-k} A_1A_{k+1}} = \frac{n-k}{k}$$

We can prove that for any k the point of intersection of the k-th order medians in an n-gon coincides with its centroid.

PROBLEM 29. Formulate the assertion of Problem 28 for $n = 4$, $k = 2$.

Answer. The line segments connecting the midpoints of the opposite sides and the midpoints of the diagonals of an arbitrary quadrilateral intersect in a single point and are bisected by this point.

The circle passing through the midpoints of three sides of a triangle (Fig. 58) is called the *Euler circle* of the triangle. It possesses a number of interesting properties (for instance, in the triangle ABC the Euler circle as well as passing through the midpoints D, E, F of its sides also passes through the feet P, Q, R of the altitudes AP, BQ and CR, and through the three points K, L, M bisecting the segments AH, BH, CH of the altitudes between the point H of their intersection and the vertices*. Therefore Euler's circle is often called the *nine-point circle* of a triangle). Since the Euler circle of the triangle ABC is circumscribed about the triangle DEF which is similar to ABC (the ratio of magnification (of similitude) being 1/2), its radius is equal to $R/2$, where R is the radius of the circumscribed circle about the initial triangle ABC. The notion of Euler's circle can be applied to any polygon inscribed in a circle in the following way.

---

* Since the quadrilateral $KFDM$ (Fig. 58) is a rectangle (because $FK \parallel BH \parallel DM$, as KF and DM are middle lines of the triangles ABH and CBH with a common base BH; $FD \parallel AC \parallel KM$, since FD and KM are middle lines of the triangles ABC and AHC with a common base AC; $BH \perp AC$), the segments FM and DK are equal and have a common midpoint. We prove in the same way that the segment EL is equal to them and its midpoint coincides with the midpoint common to FM and KD. Hence, it follows that the Euler circle, passing through the points D, E and F, also passes through K, L and M (the centre of this circle coincides with the common midpoint of DK, EL and FM, its diameter being equal to the length of these segments).

Furthermore, since, by proof, K and D are diametrically opposite points on the Euler circle and $\angle KPD = 90°$, this circle passes through the point P. In a similar way we prove that it passes through the points Q and R.

PROBLEM 30. 1°. The *Euler circle of the chord* A_1A_2 of the circle S of radius R is defined as a circle of radius $R/2$ with centre at the midpoint of the chord A_1A_2 (Fig. 59, *a*). Three Euler's circles of the sides of the triangle $A_1A_2A_3$ inscribed in the circle S intersect at one point O which is the centre of the circle of radius $R/2$ passing through the centres of the three Euler circles. This circle is called the *Euler circle of the triangle* $A_1A_2A_3$ (Fig. 59, *b*).

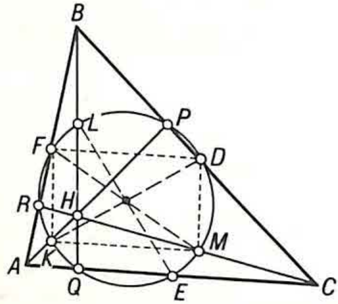


FIG. 58

2°. Suppose that we have already determined the Euler circle of the n-gon inscribed in the circle S, and it is known that its radius is equal to $R/2$ (R is the radius of the circle S). Let us now consider the $(n+1)$-gon $A_1A_2A_3 \ldots A_{n+1}$ inscribed in the circle S. In this case $n+1$ Euler's circles of the n-gons $A_2A_3 \ldots A_{n+1}$, $A_1A_3 \ldots A_{n+1}$, $\ldots$, $A_1A_2 \ldots A_n$ intersect in a single point which is the centre of the circle of radius $R/2$ passing through the centres of all the $n+1$ Euler circles. This circle is called the *Euler circle of the* $(n+1)$*-gon* $A_1A_2 \ldots A_{n+1}$ (see Fig. 59, *c* illustrating the Euler circle of a quadrilateral).

Hint. Let $A_1A_2A_3A_4$ be an arbitrary quadrilateral inscribed in the circle S. From the fact that the Euler circle of the triangle $A_1A_2A_3$ (for instance) passes through the three midpoints of the line segments H_4A_1, H_4A_2, H_4A_3 (where H_4 is the point of intersection of the altitudes $A_1A_2A_3$ (see above) it follows that it is homothetic to the circle S with the centre of similarity at the point H_4 and the ratio of magnification equal to $1/2$; therefore the midpoint of the line segment H_4A_4 belongs to this circle. Let us finally note that the midpoints of the line segments H_1A_1, H_2A_2, H_3A_3 and H_4A_4 (where H_1, H_2 and H_3 are the points of intersection of the altitudes of the corresponding triangles) coincide. It follows from the fact that the quadrilateral $A_1H_2H_1A_2$ for instance, is a parallelogram (since $A_1H_2 \parallel A_2H_1 \perp A_3A_4$ and $A_1H_2 = A_2H_1 =$ twice the distance between the centre S and A_3A_4).

Let us now assume, that for all k-gons, the number of whose sides k does not exceed $n \geqslant 4$, the existence of the Euler circle is already proved. Consider the $(n+1)$-gon $A_1A_2 \ldots A_nA_{n+1}$ inscribed in the circle S. We have to prove that the Euler circles $S_1, S_2, \ldots, S_{n+1}$ of the n-gons $A_2A_3 \ldots A_{n+1}, A_1A_3A_4 \ldots A_{n+1}, \ldots, A_1A_2 \ldots A_n$ intersect in a single point. To do this it is sufficient to prove that any three of them (say, S_1, S_2 and S_3) intersect in a single point*. Let us denote the Euler

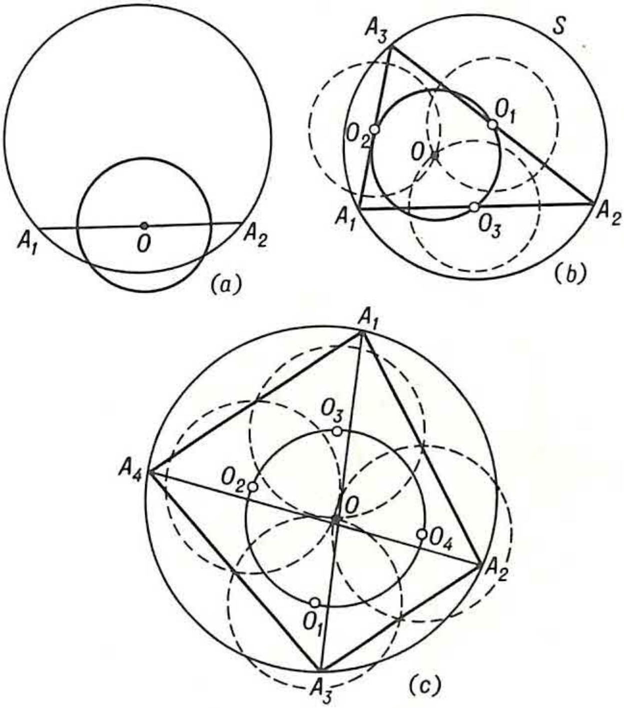


FIG. 59

circles of the $(n-1)$-gons $A_3A_4 \ldots A_{n+1}, A_2A_4A_5 \ldots A_{n+1}, A_1A_4A_5 \ldots A_{n+1}$ by S_{12}, S_{13}, S_{23}, respectively, their centres by O_{12}, O_{13}, O_{23}; the centres of the circles S_1, S_2, S_3 by O_1, O_2, O_3, and the centre of the Euler circle S_{123} of the $(n-2)$-gon $A_4A_5 \ldots A_{n+1}$ by O_{123} (see Fig. 60). As it is obvious from the figure, the triangles $O_1O_2O_3$ and $O_{23}O_{13}O_{12}$

* Since if each three of $n \geqslant 5$ circles (no two of which coincide) intersect at one point, then all the circles intersect at one point (for $n = 4$ it is already not true).

are congruent. [To prove the equality of the sides O_1O_2 and $O_{23}O_{13}$ of these triangles it is sufficient to consider the triangles $O_1O_2O_{12}$ and $O_{23}O_{13}O_{123}$ in which

$$O_{12}O_1 = O_{12}O_2 = O_{123}O_{23} = O_{123}O_{13} = \frac{R}{2},$$

$$\angle O_1O_{12}O_2 = \angle O_1O_{12}O_{123} + \angle O_{123}O_{12}O_2 =$$
$$= 2\angle O_{13}O_{12}O_{123} + 2\angle O_{123}O_{12}O_{23} = 2\angle O_{13}O_{12}O_{23}$$

and

$$\angle O_{23}O_{123}O_{13} = 2\angle O_{13}O_{12}O_{23},$$

as they are respectively an inscribed angle and an angle at the centre of the circle circumscribed about $O_{12}O_{13}O_{23}$ and are subtended by the

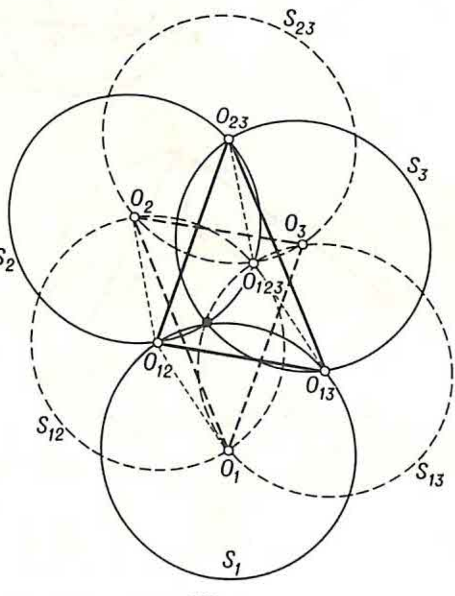


FIG. 60

same arc; we prove similarly that $O_1O_3 = O_{23}O_{12}$ and $O_2O_3 = O_{13}O_{12}$]. From the fact that $\triangle O_1O_2O_3 = \triangle O_{23}O_{13}O_{12}$ and the circles S_{23}, S_{13} and S_{12} intersect in a single point O_{123} it directly follows that the circles S_1, S_2 and S_3 also intersect at one point.

PROBLEM 31. Let $A_1A_2 \ldots A_n$ be an arbitrary n-gon inscribed in a circle S. Prove that the centroid of the n-gon (see Example 27) lies on the line segment connecting the centre of S with the centre

of the Euler circle of the n-gon and divides this segment in the ratio $(n-2):2$.

Hint. The solution of this problem can be found in [26] (see the solution of Problem 18, *c*).

PROBLEM 32. 1°. Let the triangle ABC be inscribed in the circle S and let P be an arbitrary point on this circle. Prove that the feet of the perpendiculars dropped from the point P to the sides of the triangle ABC are collinear (Fig. 61; this straight line is called the *Simson* line* of P with respect to the triangle ABC).

2°. Suppose we already know the definition of the Simson line of the point P in the circle S with respect to the n-gon $A_1A_2 \ldots A_n$ inscribed in S. Moreover, let $M \equiv A_1A_2 \ldots A_nA_{n+1}$ be an $(n+1)$-gon inscribed in S. Prove that the feet of the perpendiculars dropped from P upon $n+1$ Simson's lines of all possible n-gons formed by n vertices of the $(n+1)$-gon M are collinear. This line we shall call the *Simson line* of the point P with respect to the $(n+1)$-gon M.

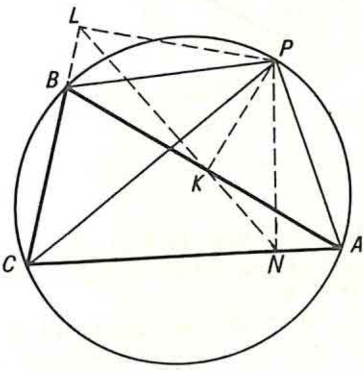


FIG. 61

Hint. The problem of the Simson line of a point P with respect to a triangle is widely known. The proof follows from the fact that, in the notation of Fig. 61, the quadrilaterals $PAMK$, $PLBK$ and $PMCL$ can be inscribed in the circle, from which it follows that $\angle AKM = \angle APM$, $\angle BKL = \angle BPL$ and $\angle MPL = 180° - \angle C = \angle APB$, whence $\angle AKM = \angle BKL$.

* Robert Simson (1687-1768), a Scottish mathematician, professor at the University of Glasgow.

A complete solution of Problem 32, and also alternative solutions of Problems 27 to 31 can be found in [27].

EXAMPLE 28. 1°. Let l_1, l_2, l_3, l_4 be four lines such that no two of them are parallel and no three of them pass through one point. Let O_1 be the centre of the circle circumscribed about the triangle formed by the straight lines l_2, l_3, l_4; O_2 the centre of the circle circumscribed about the triangle formed by the straight

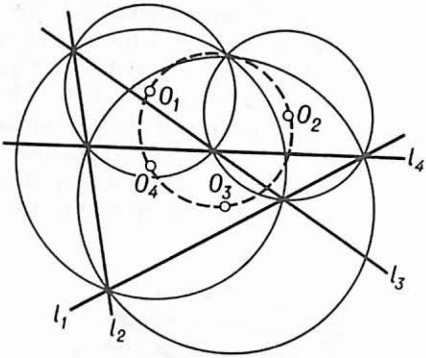


FIG. 62

lines l_1, l_3, l_4, and so on. Then the four points O_1, O_2, O_3 and O_4 lie in one circle called the *circle of the centres of the straight lines* l_1, l_2, l_3, l_4 (Fig. 62).

2°. Suppose the circle of the centres of n straight lines is already defined and suppose $n+1$ oblique lines $l_1, l_2, l_3, \ldots, l_{n+1}$ are given. Let us denote the centre of the circle of the centres of the n straight lines $l_2, l_3, \ldots, l_{n+1}$ by O_1, the centre of the circle of the centres of the n straight lines $l_1, l_3, \ldots, l_{n+1}$ by O_2 and so on. Then the $n+1$ points $O_1, O_2, O_3, \ldots, O_{n+1}$ lie on one circle called the *circle of the centres of the* $n+1$ *straight lines* $l_1, l_2, l_3, \ldots, l_{n+1}$.

Proof. 1°. Let l_1, l_2, l_3, l_4 be four oblique lines (Fig. 63), A_{12} the point of intersection of l_3 and l_4, A_{13} the point of intersection of l_2 and l_4, etc.; O_1 the centre of the circle C_1 circumscribed about the triangle formed by the straight lines l_2, l_3 and l_4, and so on. Let us first prove that the circles C_1, C_2, C_3 and C_4 intersect in one point M. In fact, if M is the point of intersection

of C_1 and C_2 other than A_{12}, then

$$\angle A_{13}MA_{12} = \angle A_{13}A_{14}A_{12} = \angle \text{ between } l_2 \text{ and } l_3;$$
$$\angle A_{12}MA_{23} = \angle A_{12}A_{24}A_{23} = \angle \text{ between } l_3 \text{ and } l_1.$$

Hence it follows that $\angle A_{13}MA_{23} = \angle$ between l_2 and $l_1 = \angle A_{13}A_{34}A_{23}$, i. e. that the circle C_3 passes through the point M. It can be proved in a similar way that C_4 also passes through M.

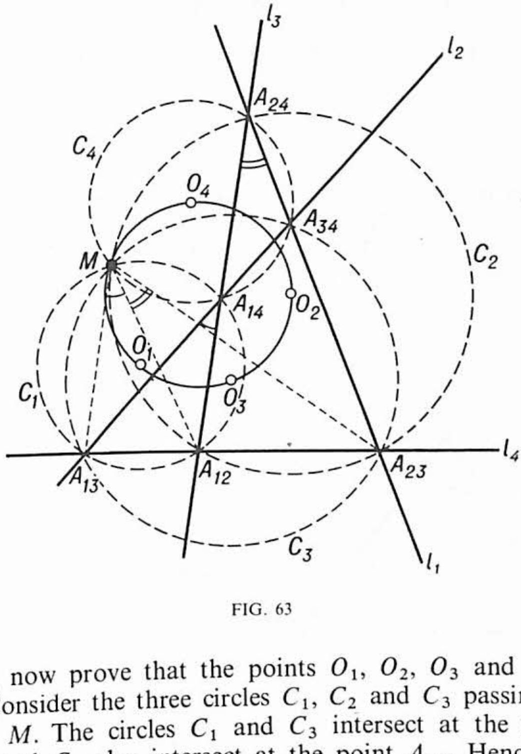


FIG. 63

We can now prove that the points O_1, O_2, O_3 and O_4 lie on a circle. Consider the three circles C_1, C_2 and C_3 passing through one point M. The circles C_1 and C_3 intersect at the point A_{13}, while C_2 and C_3 also intersect at the point A_{23}. Hence

$$\angle O_1O_3O_2 = \angle A_{13}MA_{23} = \angle A_{13}A_{34}A_{23} = \angle \text{ between } l_2 \text{ and } l_1.$$

We prove in the same way that $\angle O_1O_4O_2 = \angle$ between l_2 and $l_1 = \angle O_1O_3O_2$, from which the required assertion follows.

2°. Let us assume that for n straight lines our propositions are already proved. Then we can also consider it proved that the *arc of the circle of centres of the n straight lines* $l_1, l_2, \ldots, l_n$ *between the centres* O_1 *and* O_2 *of the circles of centres of the* $n-1$ *lines*

$l_2, l_3, \ldots, l_n$ *and the* $n-1$ *lines* $l_1, l_3, \ldots, l_n$ (respectively) *is equal to twice the angle between the straight lines* l_1 *and* l_2 (see the end of 1°). Let us now consider the $n+1$ oblique lines $l_1, l_2, \ldots, l_{n+1}$ and let O_1 be the centre of the circle of centres C_1 of the n straight lines $l_2, l_3, \ldots, l_{n+1}$ and so on, O_{12} the centre of the circle of centres C_{12} of the $n-1$ lines $l_3, l_4, \ldots, l_{n+1}$ and so on. Let us prove that the circles $C_1, C_2, \ldots, C_{n+1}$ intersect in a single point M. Indeed, let M be the point (other than O_{12}) of intersection of the circles C_1 and C_2. We then have

$$\angle O_{13}MO_{12} = \frac{1}{2} \cup O_{13}O_{12} = \angle \text{ between } l_2 \text{ and } l_3,$$

$$\angle O_{12}MO_{23} = \frac{1}{2} \cup O_{12}O_{23} = \angle \text{ between } l_3 \text{ and } l_1,$$

from which it follows that $\angle O_{13}MO_{23} = \angle$ between l_2 and $l_1 = \angle O_{13}O_{34}O_{23}$, which means that the circle C_3 passes through M. We prove in exactly the same way that each of the remaining circles $C_4, C_5, \ldots, C_{n+1}$ passes through M.

Let us now consider the three circles C_1, C_2 and C_3 passing through the single point M; C_1 and C_3 intersect also at the

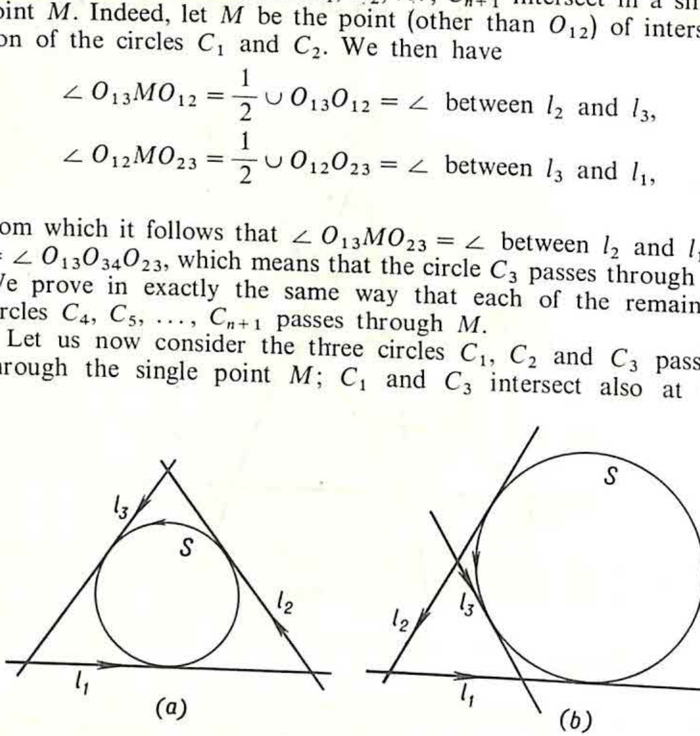


FIG. 64

point O_{13}, while C_2 and C_3 intersect at O_{23}. We have:

$$\angle O_1O_3O_2 = \angle O_{13}MO_{23} = \angle O_{13}O_{43}O_{23} = \angle \text{ between } l_2 \text{ and } l_1.$$

We show in the same way that for any point O_i ($i = 4, 5, \ldots, n+1$), $\angle O_1O_iO_2 = \angle$ between l_2 and l_1, from which it follows that all the points $O_1, O_2, O_3, O_4, \ldots, O_{n+1}$ lie on the one circle.

In Example 28 all the circumscribed circles may be replaced by inscribed ones. But there arises here an additional difficulty due

to the following circumstance: the circumcircle of the triangle (i. e. the circle passing through all of its vertices) is defined uniquely, whereas for the inscribed circle (i. e. the circle which touches its sides) we may choose (any) one of four circles – one circle touches all the sides internally while three circles touch them externally. To remove this difficulty we can proceed in the following way. Let us introduce the notion of *directed* lines and circles, indicating with an arrow the direction of motion along them. We shall then consider a directed straight line and a directed circle as *touching* each other only if their directions coincide at the point of contact. In this case we have only *a single* directed circle which touches three given directed lines l_1, l_2 and l_3, not intersecting at one point. Fig. 64 illustrates two possible cases of a directed circle inscribed in a triangle formed by l_1, l_2 and l_3: internally (a) and externally (b).

PROBLEM 33. 1°. Let l_1, l_2, l_3 and l_4 be four directed oblique lines, i. e. such that any two of them intersect and no three have a common point; let O_1, O_2, O_3 and O_4 be the centres of the incircles of the triangles formed by the lines l_2, l_3, and l_4; l_1, l_3 and l_4, and so on. Then the four points O_1, O_2, O_3 and O_4 lie on a single circle called the *circle of centres of the four directed lines* l_1, l_2, l_3 and l_4 (Fig. 65).

2°. Let the circle of centres of n directed lines be already defined and $n+1$ directed oblique lines l_1, l_2, $\ldots$, l_n and l_{n+1} be given. Let O_1 denote the centre of the circle of centres of the n lines obtained from the given $n+1$ lines by deleting l_1; O_2 the centre of the circle of centres of the n lines obtained from the given $n+1$ lines by deleting l_2, and so on. In this case the $n+1$ points O_1, O_2, $\ldots$, O_n, O_{n+1} lie on a circle called the *circle of centres of the* $n+1$ *directed lines* l_1, l_2, $\ldots$, l_n, l_{n+1}.

Hint. See the solution of Example 28.

PROBLEM 34. The definition of the *orthocentre of a polygon inscribed in a circle*.

1°. As is well known, the point of intersection of the three altitudes of a triangle is called the *orthocentre of the triangle*.

2°. Suppose the orthocentre of the n-gon $A_1A_2 \ldots A_n$ inscribed in a circle S is already defined and suppose we have an $(n+1)$-gon $A_1A_2 \ldots A_nA_{n+1}$ inscribed in S. Let us then denote the orthocentres of the $n+1$ polygons $A_2A_3 \ldots A_{n+1}$, $A_1A_3A_4 \ldots A_{n+1}$, $\ldots$, $A_1A_2 \ldots A_n$ by H_1, H_2, $\ldots$, H_{n+1}, respectively. In this case the circles with the same radii as S having the points H_1, H_2, $\ldots$, H_{n+1}

as centres intersect in a single point H, which is called the *orthocentre* of the $(n + 1)$-gon $A_1A_2 \ldots A_{n+1}$ (e. g. Fig. 66 illustrates the orthocentre of the quadrilateral $A_1A_2A_3A_4$). The orthocentres of the polygons inscribed in a circle have a number of properties similar to those of the orthocentres of triangles. We are not going to list these properties here. They, of course, must be proved by

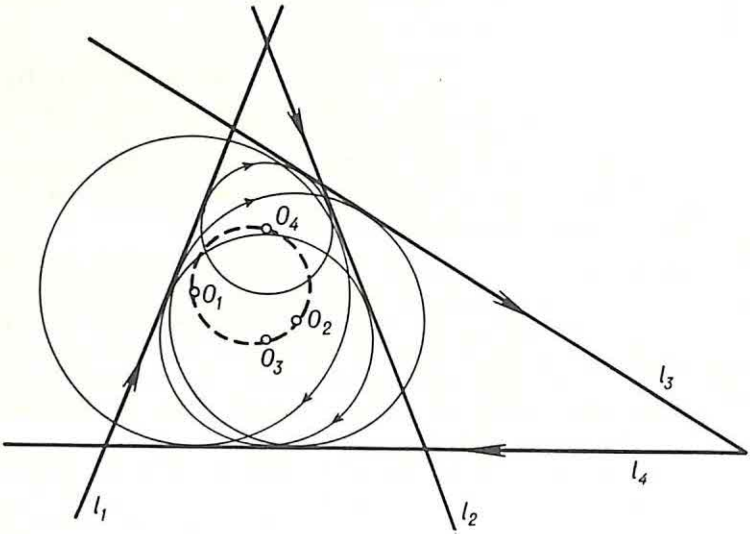


FIG. 65

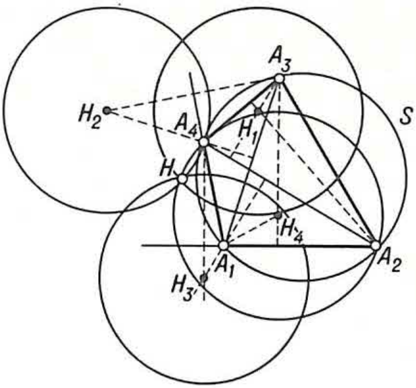
the method of mathematical induction (since the orthocentre of a polygon is determined by induction).

Hint. The solution of Problem 34 can be found in [27].

PROBLEM 35. 1°. The *central point of two* (intersecting) *straight lines* is defined as the point of their intersection (Fig. 67, *a*).

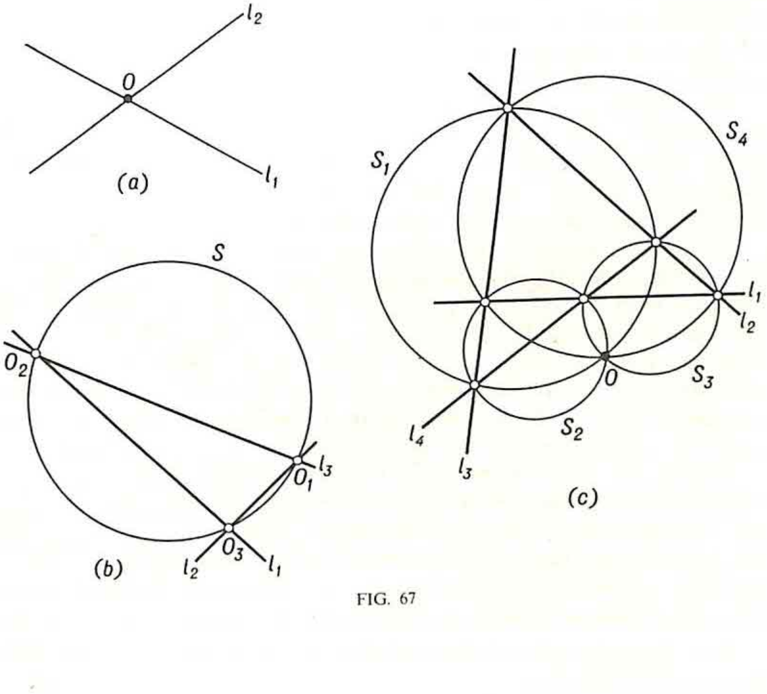
The *central circle of three oblique straight lines* (see Example 28) is defined as a circle passing through the central points of each pair of these lines (Fig. 67, *b*).

Let there now be given four oblique lines l_1, l_2, l_3, l_4. Let us denote the central circle of the three lines l_2, l_3, l_4 by S_1, the central circle of the three lines l_1, l_3, l_4 by S_2 and so on. Then the four circles S_1, S_2, S_3, S_4 intersect in a single point O which is called the *central point of the four straight lines* l_1, l_2, l_3, l_4 (Fig. 67, *c*).



A_3
H_2
A_4
H_1
S
H
H_4
A_1
A_2
H_3

FIG. 66



l_2
O
l_1
(a)
S
O_2
O_1
l_3
O_3
l_2
l_1
(b)
S_4
S_1
l_1
l_2
O
S_3
l_4
S_2
l_3
(c)

FIG. 67

2°. Suppose that the central circle of $2n-1$ straight lines and the central point of $2n$ straight lines have already been defined, and suppose the $2n+1$ oblique lines $l_1, l_2, \ldots, l_{2n}, l_{2n+1}$ are given. Let us denote the central point of the $2n$ lines $l_2, l_3, \ldots, l_{2n}, l_{2n+1}$ by A_1, the central point of the $2n$ lines $l_1, l_3, \ldots, l_{2n}, l_{2n+1}$ by A_2, and so on; the central point of the $2n$ lines $l_1, l_2, \ldots, l_{2n}$ by A_{2n+1}. Then the points $A_1, A_2, \ldots, A_{2n+1}$ lie on a single circle which we shall call the *central circle of the* $2n+1$ *lines* $l_1, l_2, \ldots, l_{2n}, l_{2n+1}$.

Finally, suppose $2n+2$ oblique lines $l_1, l_2, \ldots, l_{2n+1}, l_{2n+2}$ are given. Let us denote the central circle of the $2n+1$ lines $l_2, l_3, \ldots, l_{2n+1}, l_{2n+2}$ by S_1, the central circle of the $2n+1$ lines $l_1, l_3, \ldots, l_{2n+1}, l_{2n+2}$ by S_2, and so on; and denote the central circle of the $2n+1$ lines $l_1, l_2, \ldots, l_{2n+1}$ by S_{2n+2}. Then the circles $S_1, S_2, \ldots, S_{2n+1}, S_{2n+2}$ intersect at a point which we shall call the *central point of the* $2n+2$ *lines* $l_1, l_2, \ldots, l_{2n+1}, l_{2n+2}$.

Hint. In solving this problem it is advisable to make use of inversion properties. See, for instance, [7], [11], [16].

PROBLEM 36. 1°. Given: three oblique lines l_1, l_2, l_3. The centre of the circle circumscribed about the triangle which they form is called the *central point of the three lines.*

Let us now consider four oblique lines l_1, l_2, l_3, l_4. We denote the central point of the three lines l_2, l_3, l_4 by A_1, the central point of the three lines l_1, l_3, l_4 by A_2, and so on. Then the four points A_1, A_2, A_3, A_4 lie on one circle (see Example 28) which is called the *central circle of the four lines* l_1, l_2, l_3, l_4.

2°. Let the central point of $2n-1$ lines and the central circle of $2n$ lines be already defined, and let there be given $2n+1$ oblique lines $l_1, l_2, \ldots, l_{2n}, l_{2n+1}$. We denote the central circle of the $2n$ lines $l_2, l_3, \ldots, l_{2n}, l_{2n+1}$ by S_1, the central circle of the $2n$ lines $l_1, l_3, \ldots, l_{2n}, l_{2n+1}$ by S_2, and so on, the central circle of the $2n$ lines $l_1, l_2, \ldots, l_{2n}$ by S_{2n+1}. Then the circles $S_1, S_2, \ldots, S_{2n+1}$ intersect in a single point which we shall call the *central point of the* $2n+1$ *lines* $l_1, l_2, \ldots, l_{2n}, l_{2n+1}$.

Finally, let there be given $2n+2$ oblique lines $l_1, l_2, \ldots, l_{2n+2}$. We shall denote the central point of the $2n+1$ lines $l_2, l_3, \ldots, l_{2n+2}$ by A_1, the central point of the $2n+1$ lines $l_1, l_3, \ldots, l_{2n+2}$ by A_2, and so on, the central point of $2n+1$ lines $l_1, l_2, \ldots, l_{2n+1}$ by A_{2n+2}. Then the points $A_1, A_2, \ldots, A_{2n+2}$ all lie on a circle which we shall call the *central circle of the* $2n+2$ *lines* $l_1, l_2, \ldots, l_{2n+2}$.

Hint. The propositions formulated here can be proved in the same way as those of Problem 35.

The pair formed by taking together a point A and a direction specified by a straight line a, passing through the point, is called a *linear element*, usually denoted by (A, a). We shall call n linear elements (A_1, a_1), (A_2, a_2), $\ldots$, (A_n, a_n) *concyclic* if the straight lines a_1, a_2, $\ldots$, a_n are oblique (i. e. no two are parallel and no three meet in a single point) (see Example 28) and the n points A_1, A_2, $\ldots$, A_n lie on one circle.

PROBLEM 37. 1°. Given: two linear elements (A_1, a_1) and (A_2, a_2), such that A_1 differs from A_2, and the straight lines a_1 and a_2 intersect. The circle passing through the points A_1, A_2 and through the point of intersection of a_1 and a_2 is called the *directing circle of the two linear elements* (Fig. 68, *a*). The directing circles of three pairs of linear elements (A_1, a_1) and (A_2, a_2), (A_1, a_1) and (A_3, a_3), (A_2, a_2) and (A_3, a_3) (such that all the points A_1, A_2, A_3 differ, and a_1, a_2, a_3 are oblique lines) intersect in a point called the *directing point of the three linear elements* (A_1, a_1), (A_2, a_2) and (A_3, a_3) (Fig. 68, *b*).

2°. Suppose the directing circle of $2n-2$ concyclic linear elements and the directing point of $2n-1$ concyclic linear elements has

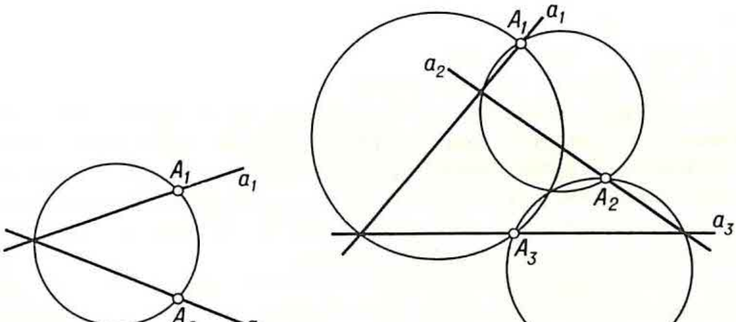


FIG. 68

already been defined by us. Consider $2n$ concyclic linear elements. In this case it is clear that each set of $2n-1$ linear elements has its own directing point. The $2n$ directing points formed from the $2n$ sets of $2n-1$ elements all lie on a circle, called the *directing circle of the* $2n$ *concyclic linear elements*. Furthermore, if we consider $2n+1$ concyclic linear elements, then the $2n+1$ sets formed by taking all the possible sets of $2n$ linear elements from the $2n+1$

linear elements determine $2n+1$ directing circles, all of which intersect in a point which is called the *directing point of the* $2n+1$ *concyclic linear elements.*

Hint. See the hint to Problem 35.

Sec. 6. Induction on the Number of Dimensions

When studying a course of three-dimensional geometry we cannot help noticing the well-known analogy between three- and two-dimensional theorems. Thus, the properties of the parallelepiped are in many ways similar to those of the parallelogram (compare, for instance, the following theorems: "The opposite faces of a parallelepiped are equal and its diagonals intersect in a single point and are bisected by this point" and "The opposite sides of a parallelogram are equal and its diagonals are bisected by the point of their intersection"), and the properties of a sphere are similar to those of a circle (compare, for instance, the following two theorems: "A tangent plane to a sphere is perpendicular to the radius passing through the point of contact" and "A tangent line to a circle is perpendicular to the radius passing through the point of contact"). But at the same time there is an essential difference between the properties of plane and space figures. The principal difference is that figures in a plane have two dimensions ("length" and "width"), whereas solids have three dimensions ("length", "width" and "altitude" (or "height")). Accordingly, the position of a point in a plane is completely determined by two coordinates x and y (Fig. 69, *b*), whereas to determine the position of a point in space we have to know three coordinates x, y and z (Fig. 69, *c*). Therefore, ordinary space we often call *three-dimensional space*, ("the space of three dimensions") while the plane is said to be a *two-dimensional space* ("the space of two dimensions").

This terminology can be extended. The position of a point on a straight line is completely determined by a single coordinate x (Fig. 69, *a*). This is due to the fact that on a straight line all figures (segments) have only one dimension ("length"). Therefore a straight line is said to be a *one-dimensional space*. This enables us to say that the number of dimensions of a space may be equal to one, two, or three.

Three-dimensional theorems are usually more complicated than the corresponding two-dimensional propositions, while the proper-

ties of plane (two-dimensional) figures, are, of course, much more complicated than those of figures on a straight line (segments). Many proofs of three-dimensional theorems are based on a knowledge of the corresponding two-dimensional propositions. For instance, the proof of the theorem that the diagonals of a parallelepiped are bisected by their point of intersection makes use of the corresponding property of the diagonals of a parallelogram.

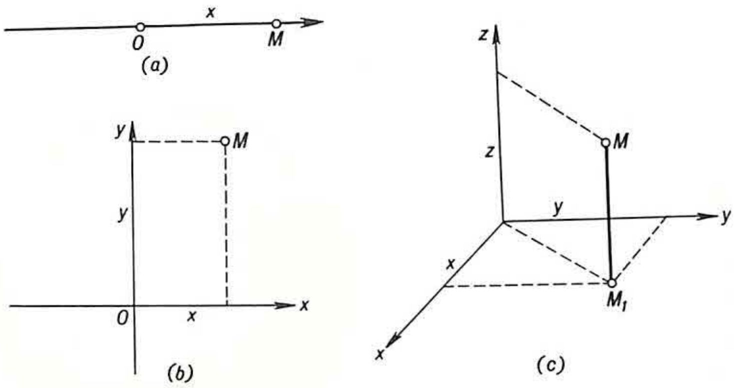


FIG. 69

Again, proofs of "two-dimensional" theorems are sometimes based on the analogous "one-dimensional" theorems. This makes it possible in some geometric problems to use *induction on the number of dimensions*. By this is meant a successive passage from one to two- and then to three-dimensional space. Some examples of this kind are just considered in the present section. Induction on the number of dimensions is often used simultaneously with a more usual application of the method of mathematical induction, and sometimes it may be replaced by the latter. When working through the examples and solving the problems of this section, the reader should bear in mind that in space the sphere (the spherical surface, Fig. 70, *c*) corresponds to the circle in a plane (as the locus of points equidistant from a given point O, Fig. 70, *b*), while on a straight line (Fig. 70, *a*) there corresponds the pair of points equidistant from a given point O. That is, a circle in a plane corresponds to a sphere in space and to a segment on a straight line. Finally, a triangle ABC in the plane (Fig. 71, *b*) corresponds to a tetrahedron in space (a tetrahedron is an arbitrary triangular pyramid having four vertices A, B, C, D, Fig. 71, *c*) and to

a segment AB having two "vertices" A and B on a straight line (Fig. 71, *a*).

It should be noted here that the question as to what three-dimensional proposition corresponds to a given two-dimensional theorem, generally speaking, cannot be answered uniquely. It is sometimes convenient to suppose that, corresponding to a triangle in the plane, we have in space not a tetrahedron, i. e. a three-

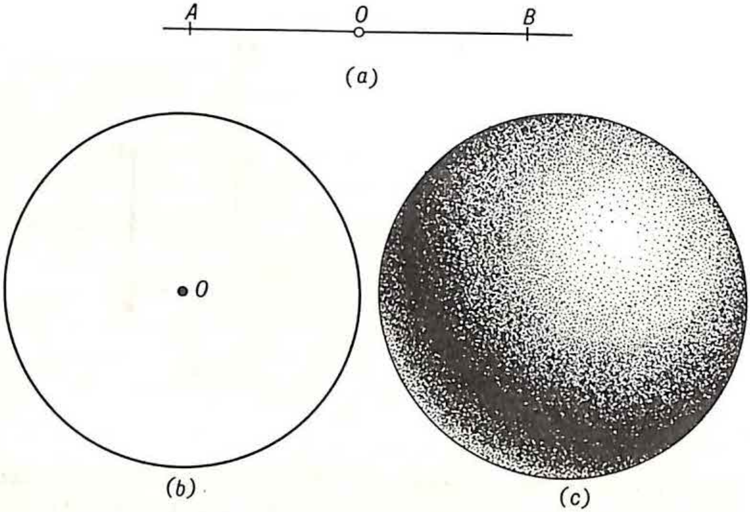


FIG. 70.

dimensional figure, a figure with one more dimension than the triangle, but the same triangle only located in space. Analogously, we can suppose that a straight line in a plane corresponds to either a straight line or a plane in space. In this way we can obtain various "three-dimensional analogues" of a given two-dimensional theorem. For instance, the theorem "The sum of the squares of the distances from a point M in a plane to the vertices of a regular n-gon (with O as centre), inscribed in a circle of radius R, is equal to $n(R^2 + OM^2)$" (see Problem 234 in [21]) is related to the following two three-dimensional theorems: "The sum of the squares of the distances from a point M in space to the vertices of a regular n-gon with centre O, inscribed in a circle of radius R, is equal to $n(R^2 + OM^2)$" and "The sum of the squares of the distances from a point M in space to the

vertices of a regular polyhedron with n vertices, inscribed in a sphere of radius R with O as centre, is equal to $n(R^2 + OM^2)$". Both theorems are true and are generalizations of the corresponding "two-dimensional" theorem. Thus, in deducing them we use induction on the number of dimensions. We are not going to consider this problem in detail, leaving the reader a chance to do some independent work. We suggest that for this he compares

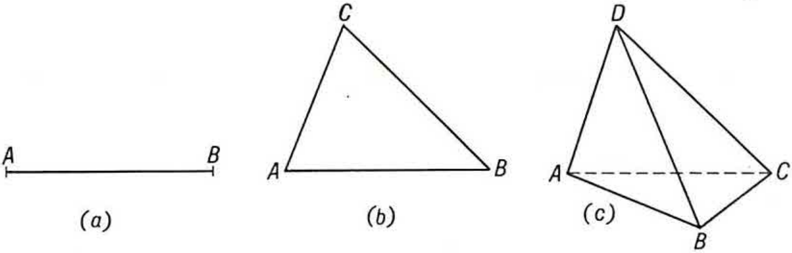


FIG. 71

the passage from "one-dimensional" theorem to the "two-dimensional" and "three-dimensional" cases, for instance, in Examples 30 and 38 on the one hand and in 36 and 37 on the other.

In modern mathematics, both in the theoretical and applied fields, the notion of an *n-dimensional space* plays a very important role. It appears in the following way. A one-dimensional space (a straight line), a two-dimensional space (a plane) and a three-dimensional space can be characterized in the following way: an arbitrary point in one-dimensional space may be specified by one number, in two-dimensional space it is specified by two numbers, while in three-dimensional space a point is specified by three numbers. These numbers are called the coordinates of the point (see Fig. 69, a, b and c). But in mathematics itself, as well as in other sciences, we often come across objects specified by a certain number n of "parameters" (often exceeding three), describing the given object. For instance, a sphere $\sum$ is specified in space by the coordinates (x, y, z) of its centre Q and its radius r (Fig. 72, a), and a straight line l in space, say, by giving the coordinates (x, y) of the point P, where it pierces the plane xOy, and of the point Q (x_1, z) where it intersects the plane xOz (Fig. 72, b). Accordingly, we can speak of a *four-dimensional* space of spheres $\sum (x, y, z, r)$ and of a *four-dimensional* space of straight lines l (x, y, x_1, z). Analogously, in physics, for instance, the state

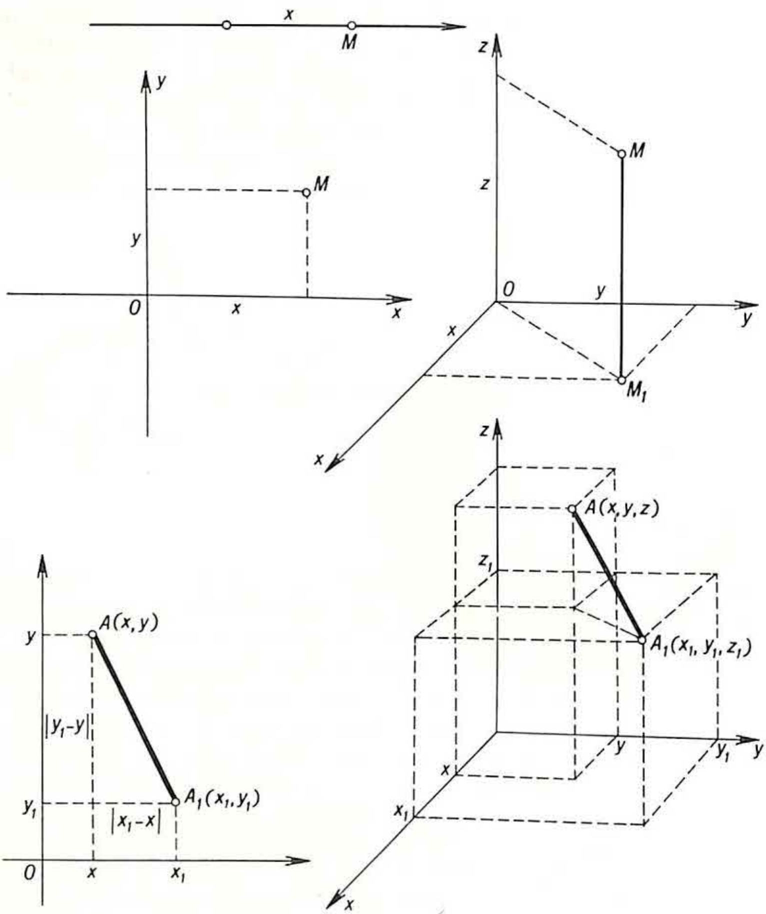


FIG. 72

of a material point M (moving in space) is specified by comparing M with the point $\mathbf{M}$ of a *six-dimensional* phase space with the coordinates $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ equal to the coordinates x, y, z of the point M and to the projections $\dot{x}$, $\dot{y}$, $\dot{z}$ of the velocity vector $\mathbf{v}$ of this point on the coordinate axes (the velocity components of the point M along the directions of the x-, y- and z-axis). In general, a set of objects each of which is specified by n numerical parameters that single out an individual object Φ from the totality, is called an *n-dimensional space*, and the parameters

$x_1, x_2, \ldots, x_n$ are called the *coordinates* of Φ (which is often written in the following way: $\Phi = \Phi(x_1, x_2, \ldots, x_n)$). The objects Φ are considered as the "points" of an n-dimensional space with coordinates $x_1, x_2, \ldots, x_n$.

Suppose now we have an n-dimensional space. It is often convenient to introduce to the space a *distance* between points, by taking the distance d_{MN} between the points $M(x_1, x_2, \ldots, x_n)$ and $N(y_1, y_2, \ldots, y_n)$ to be equal to

$$d_{MN} = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2}. \qquad (**)$$

This formula is justified because in the case of points on a straight line, in a plane and in ordinary three-dimensional space the distance between the points $M(x)$ and $N(x_1)$, $M(x, y)$ and $N(x_1, y_1)$, $M(x, y, z)$ and $N(x_1, y_1, z_1)$ is respectively equal to $|x_1 - x|$ $(= \sqrt{(x_1 - x)^2})$, $\sqrt{(x_1 - x)^2 + (y_1 - y)^2}$ and $\sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2}$ (see Fig. 69, *a*, *b*, *c*; note that in Fig. 69, *b* the legs MP and NP of the right-angled triangle MNP are equal to $|x_1 - x|$ and $|y_1 - y|$, and in Fig. 69, *c* the segment NM_1 can be proved to be equal to $\sqrt{(x_1 - x)^2 + (y_1 - y)^2}$, while $MM_1 = |z_1 - z|$). The distance d_{MN} defined by the formula (**) has the property that $d_{MN} = 0$ implies that the points M and N coincide. Also, when the distance d_{MN} is small, all the differences $|y_1 - x_1|$, $|y_2 - x_2|$, $|y_3 - x_3|$, and so on must be small, i.e. that the object characterized by the point M is close to the object characterized by the point N in all parameters ("in all respects" so to speak). (For instance, a small distance $d_{\Sigma\Sigma_1}$ between the spheres $\sum(x, y, z, r)$ and $\sum_1(x_1, y_1, z_1, r_1)$ in a four-dimensional space of spheres means that the centre $Q(x, y, z)$ of the sphere $\sum$ is near to the centre $Q_1(x_1, y_1, z_1)$ of the sphere $\sum_1$, and the radius r of the sphere $\sum$ differs little from the radius r_1 of the sphere $\sum_1$).

Furthermore, an n-dimensional space in which the distance d_{MN} between points $M(x_1, x_2, \ldots, x_n)$ and $N(y_1, y_2, \ldots, y_n)$ is determined by the formula (**), is called an *n-dimensional Euclidean space*. Many notions taken from ordinary school geometry are valid for n-dimensional Euclidean geometry. For instance, given the distances d_{AB}, d_{BC} and d_{AC} between three points A, B, C we are able to determine the angle BAC by the cosine law: $d_{BC}^2 = d_{AB}^2 + d_{AC}^2 - 2d_{AB} \cdot d_{AC} \cos \angle ABC$; we may write $AB \perp AC$ if this angle is a right angle, i.e. if the Pythagorean theorem $(d_{BC}^2 = d_{AB}^2 + d_{AC}^2)$ is satisfied for the triangle ABC. In this little elementary book it is not possible for us to discuss this important notion fully

(for further details see [13], [22], [23]; note, by the way, that the last two books, as well as the concluding chapter of [6], are devoted to the concept of *four-dimensional* space).

All the examples in this section may be considered as "n-dimensional cases". We strongly recommend the reader not to confine himself to three-dimensional versions only of the propositions in question, but to try step beyond the barrier $n = 3$ and consider the propositions independently for $n > 3$ (where n is the number of dimensions). Only such a step justifies here item 2° of the method of mathematical induction, i. e. the passage from a definite but *arbitrary* value of n to the next value $n + 1$, whereas below we demonstrate not "complete induction" but only the passage from one theorem to another related one (say, from a two-dimensional theorem to a three-dimensional). Strictly speaking, we have no grounds even for referring to the method of mathematical induction.

Let us point out here that in defining the notions of three-dimensional and n-dimensional geometry the method of induction plays a very important role. Thus, the very definition (or description) of *polyhedra* as solids bounded by polygons (and polygons

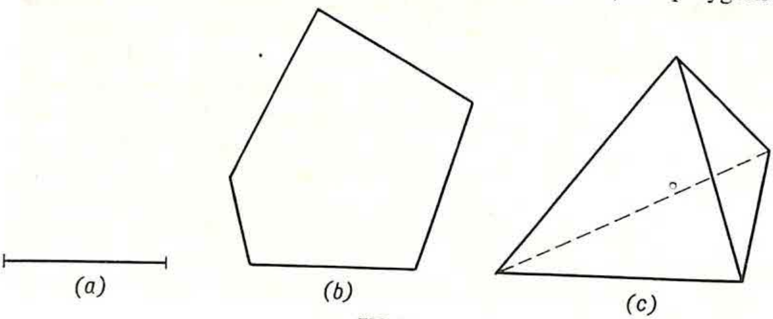


FIG. 73

as figures bounded by line segments which in turn play the part of "polygons" in one-dimensional geometry, cf. Fig. 73, a, b, c) may be considered as based on induction on the number of dimensions*.

* This is all the more true when we speak of the definition and theory of *regular polyhedra* bounded by equal plane regular polygons (the n-dimensional analogues of regular polyhedra are called *regular polytopes*). However, no kind of induction can explain why there exists only one type of "regular polytope" (i. e. a line segment) on a straight line, ∞ types in a plane (regular k-gons exist for all $k \geqslant 3$), 5 types in a three-dimensional space, 6 types in a four-dimensional space, 3 types in an n-dimensional space, where $n \geqslant 5$ (for more detail see, for instance, [5]).

In some cases induction on the number of dimensions plays an important part in constructing objects in n-dimensional geometry. Thus, the *line segment* AB (Fig. 74, a) is a "one-dimensional analogue of the square" (the line segment is the only connected figure in one-dimensional geometry). Erecting at its end-points A and B perpendiculars to AB and laying off on them segments AA_1 and BB_1 equal to AB, we obtain a *square* ABB_1A_1 (Fig. 74, b). Then laying off (on one side of the square) line segments $AA^{(1)}$,

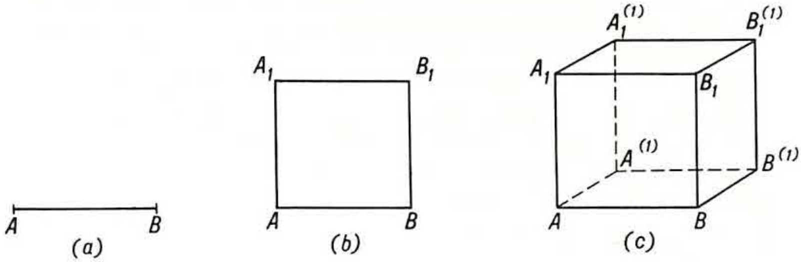


FIG. 74

$BB^{(1)}$, $A_1A_1{}^{(1)}$ and $B_1B_1{}^{(1)}$ (each equal to AB), where $AA^{(1)} \perp AB$, $AA^{(1)} \perp AA_1$ and so on, we come to a *cube* $ABB_1A_1A^{(1)}B^{(1)}B_1^{(1)}A_1^{(1)}$ (Fig. 74, c). Given a three-dimensional cube, we can construct a *four-dimensional* cube using the same method, and so on. In many cases the essence of the notions and theorems of n-dimensional geometry becomes clearer by considering first their two-dimensional and three-dimensional (and sometimes even one-dimensional) variants. This justifies the application here of the method of induction on the number of dimensions.

However, it may so happen that a certain situation appears quite differently in the three-dimensional and many-dimensional cases or in the two- and three-dimensional cases, and so the method of induction will not work. For instance, the "one-dimensional counterpart" of the very complicated two-dimensional problem of colouring geographic maps, considered in detail in Section 2, is of no interest: a "map" on a straight line can only be understood as a division of the whole line or part of it into nonoverlapping segments, the "countries" of a one-dimensional map. It is clear that if each "country" consists of one (connected) piece, then such a "map" can obviously always be coloured with *two* colours in such a way that neighbouring countries are coloured with different colours. On the other hand, in three-dimensional or

many-dimensional space it is quite simple to select *any* number of (say, polyhedral) "countries" or domains of space, *any two* of which are adjacent to each other. It is sufficient to consider, say, N elongated "bars" placed side by side (from the 1st to the N-th) and N similar bars placed perpendicularly on them. We then suppose that a "country" is formed by the i-th lower and i-th upper bars together, where $i = 1, 2, \ldots, N$ (Fig. 75). Thus, for

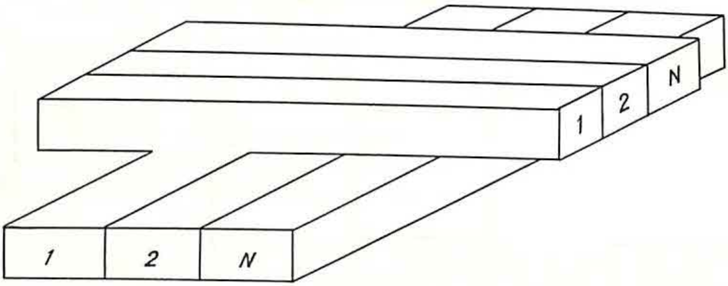


FIG. 75

a "proper colouring of maps" in space *any* – arbitrary large – number of colours may be necessary!

In conformity with the general plan of the book, the present section consists of four parts: (1) calculation by induction on the number of dimensions; (2) proof by induction on the number of dimensions; (3) finding loci by induction on the number of dimensions; (4) definition by induction on the number of dimensions. We shall not consider solving constructional problems by induction on the number of dimensions here, since the idea of construction in space is not sufficiently definite. In all cases the three-dimensional proposition is considered to be the main one, though the corresponding result in the two-dimensional case is often the most interesting. In these cases the passage from the two-dimensional to the three-dimensional case is only outlined and is not carried out in its entirety.

1. Calculation by Induction on the Number of Dimensions

EXAMPLE 29. Into how many parts is space divided by n planes, each three of which intersect and no four of which have a common point? (We shall call such planes oblique planes).

Let us consider in succession the following problems.

A. Into how many parts is a straight line divided by n points?

Solution. Let us denote the number of parts by $F_1(n)$. Then, obviously, $F_1(n) = n + 1$.

B. Into how many parts is a plane divided by n straight lines, each two of which intersect and no three of which have a common point (i. e. by n oblique lines)?

Solution. 1°. One line breaks a plane into two pieces.

2°. Suppose that we know the number $F_2(n)$ of parts into which n oblique lines divide the plane and consider $n + 1$ oblique lines. The first n of them divide the plane into $F_2(n)$ parts; the $(n + 1)$-th line l, by hypothesis, intersects each of the remaining n lines at n different points. The latter points divide the line l into $F_1(n) = n + 1$ parts (see item A). Consequently, the line l cuts across $n + 1$ of the previously obtained parts of the plane and hence adds $F_1(n) = n + 1$ new parts to the $F_2(n)$ parts. Thus,

$$F_2(n + 1) = F_2(n) + F_1(n) = F_2(n) + (n + 1). \tag{13}$$

Replacing n in equality (13) by the values $n - 1, n - 2, \ldots, 2, 1$, we get:

$$\begin{aligned} F_2(n) &= F_2(n - 1) + n, \\ F_2(n - 1) &= F_2(n - 2) + (n - 1), \\ &\cdots\cdots\cdots\cdots \\ F_2(3) &= F_2(2) + 3, \\ F_2(2) &= F_2(1) + 2. \end{aligned}$$

Let us add the equalities. Since $F_2(1) = 2$, we have:

$$\begin{aligned} F_2(n) = F_2(1) + [n + (n - 1) + \ldots + 2] = \\ = 1 + [n + (n - 1) + \ldots + 2 + 1], \end{aligned}$$

and finally:

$$F_2(n) = 1 + \frac{n(n + 1)}{2} = \frac{n^2 + n + 2}{2}$$

(see formula (2) of Introduction).

C. Into how many parts is space divided by n planes each three of which intersect and no four of which have a common point?

Solution. 1°. One plane divides space into two parts.

2°. Suppose that we know the number $F_3(n)$ of parts into which space is divided by n oblique planes, and let us consider $n + 1$

oblique planes. The first n of them divide space into $F_3(n)$ parts. These n planes intersect the $(n+1)$-th plane π along n oblique lines, and, consequently, divide it into $F_2(n) = \frac{n^2+n+2}{2}$ parts (see item B). Hence, we get the following relation

$$F_3(n+1) = F_3(n) + F_2(n) = F_3(n) + \frac{n^2+n+2}{2}. \qquad (14)$$

Replacing n by $n-1$, $n-2$, ..., 2, 1, we have

$$F_3(n) = F_3(n-1) + \frac{(n-1)^2+(n-1)+2}{2},$$

$$F_3(n-1) = F_3(n-2) + \frac{(n-2)^2+(n-2)+2}{2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_3(3) = F_3(2) + \frac{2^2+2+2}{2},$$

$$F_3(2) = F_3(1) + \frac{1^2+1+2}{2}.$$

Adding all these equalities, we obtain:

$$F_3(n) = F_3(1) + \frac{1}{2}[(n-1)^2 + (n-2)^2 + \ldots + 1^2] + \\ + \frac{1}{2}[(n-1) + (n-2) + \ldots + 1] + \frac{1}{2}\underbrace{[2+2+\ldots+2]}_{n-1 \text{ times}},$$

and finally, taking into account formulas (2) and (3) of the Introduction and the fact that $F_3(1) = 2$:

$$F_3(n) = 2 + \frac{n(n-1)(2n-1)}{12} + \frac{(n-1)n}{4} + (n-1) = \\ = \frac{(n+1)(n^2-n+6)}{6}.$$

PROBLEM 38. Into how many parts is space divided by n spheres, each two of which intersect each other?

Consider successively the following problems.

A. Into how many parts is a straight line divided by n "one-

dimensional circles", i. e. by n pairs of points (see the introductory part to this section).

Answer. $2n$ points divide a straight line into $2n+1$ parts.

A'. Find the number $\Phi_1(n)$ of parts into which a circle is divided by n pairs of points situated on this circle.

Answer. $\Phi_1(n) = 2n$.

B. Find the number $\Phi_2(n)$ of parts into which a plane is divided by n pairwise intersecting circles.

Solution. Since n circles intersect an $(n+1)$-th circle at n pairs of points, and consequently divide it into $\Phi_1(n) = 2n$ parts (see A'), the $(n+1)$-th circle intersects $\Phi_1(n) = 2n$ of the $\Phi_2(n)$ parts into which the n circles divide the plane. Hence, we get the following equality

$$\Phi_2(n+1) = \Phi_2(n) + \Phi_1(n) = \Phi_2(n) + 2n.$$

Using this equality and taking into account that $\Phi_2(1) = 2$, we have:

$$\Phi_2(n) = n^2 - n + 2.$$

B'. Into how many parts is a sphere divided by n pairwise intersecting circles located on this sphere?

Answer. Into $\Phi_2(n) = n^2 - n + 2$ parts.

C. Into how many parts is space divided by n spheres, each two of which intersect each other? (The initial problem).

Solution. Since n spheres intersect an $(n+1)$-th sphere along n circles and, consequently, break its surface into $\Phi_2(n) = n^2 - n + 2$ parts (see B'), then if n pairwise intersecting spheres break space into $\Phi_3(n)$ parts, $n+1$ spheres will divide it into $\Phi_3(n+1) = \Phi_3(n) + \Phi_2(n) = \Phi_3(n) + (n^2 - n + 2)$ parts. So, taking into account that $\Phi_3(1) = 2$, we find that

$$\Phi_3(n) = \frac{n(n^2 - 3n + 8)}{3}.$$

2. Proof by Induction on the Number of Dimensions

EXAMPLE 30. A tetrahedron whose four vertices are numbered *1*, *2*, *3* and *4* is broken into smaller tetrahedra so that each two of the tetrahedra obtained have either no points in common at all, or a common vertex, or a common edge (but not a part of an edge), or a common face (but not a part of a face). All the vertices of the newly obtained smaller tetrahedra are numbered

by the same numbers *1*, *2*, *3*, *4* in the following way: all the vertices lying on a face of the large tetrahedron are numbered with the three numbers with which the vertices of this face are numbered, and all the vertices lying on some edge of the large tetrahedron are numbered with the two numbers with which the end-points of this edge are numbered. Prove that there can be found at least one smaller tetrahedron, all the four vertices of which are numbered with different numbers.

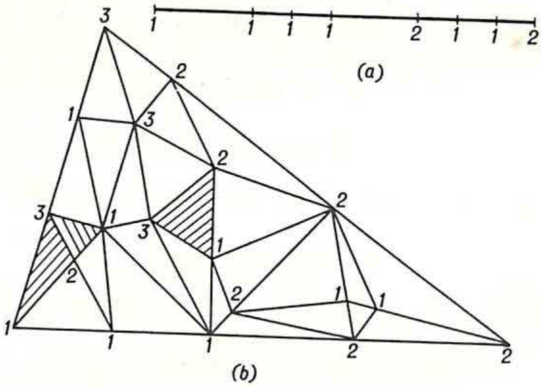


FIG. 76

Consider successively the following problems.

A. A line segment whose end-points are denoted by *1* and *2* is divided into several nonintersecting smaller segments and all points of division are numbered *1* or *2* (Fig. 76, *a*). Prove that there can be found at least one segment of the subdivision whose end-ponts are numbered with different numbers.

Solution. Let us show that the number of segments denoted *1 2* is *odd*, from which it follows that there exists at least one such segment (since zero is an even number). Let us denote by A the number of end-points of the segments of the subdivision numbered by the number *1*. This number will obviously be odd, since each number *1* found inside the given (larger) segment (we shall denote the number of such numbers *1* by k) is an end-point of two segments of the subdivision, and only one number *1*, with which the end-point of the given segment is numbered, belongs to a single segment of the subdivision. Hence

$$A = 2k + 1.$$

On the other hand, let p be the total number of segments *1 1* in our division, and q the total number of segments *1 2*. Then the number A of vertices *1* will equal

$$A = 2p + q.$$

From the equality

$$2k + 1 = 2p + q$$

it follows that q is odd.

B. A triangle whose vertices are numbered *1*, *2* and *3* is divided into smaller triangles so that any two triangles obtained have either no common points at all, or a common vertex, or a common side (but not a part of a side!). All the vertices of the smaller triangles are numbered with *1*, *2* or *3*, the vertices lying on the side of the original triangle are numbered with one of the numbers by which the end-points of this side are denoted (Fig. 76, *b*). Prove that there can be found at least one triangle of the subdivision all of whose vertices are numbered with different numbers.

Solution. Let us show that the number of triangles *1 2 3* is odd. To do this, let us count the total number A of sides of the triangles of the subdivision denoted *1 2*. Let us now denote the number of segments *1 2* lying inside the large triangle by k, and the number of such segments found on the side *1 2* of the large triangle by l (no segments *1 2* whatsoever can be found on the remaining two sides of the large triangle). Since each of the first k segments of the subdivision belongs to two small triangles, and each of the last l segments to one small triangle, we have

$$A = 2k + l.$$

On the other hand, let p denote the number of small triangles of the subdivision whose vertices are numbered *1 2 2* or *1 2 1*, and q the number of triangles *1 2 3*. Since each of the first p triangles has two sides *1 2*, and each of the last q triangles one such side, we have

$$A = 2p + q.$$

From the equality

$$2k + l = 2p + q$$

it follows that q is even or odd simultaneously with l. But the number l is odd by virtue of the proposition A, hence q is also odd.

C. The proposition formulated at the beginning of this example.

Solution. Let A be the number of faces of the tetrahedra of the subdivision denoted by *1 2 3*. If k such faces lie inside the original tetrahedron and l faces on its face *1 2 3*, then

$$A = 2k + l.$$

On the other hand, if p denotes the number of small tetrahedra numbered *1 1 2 3*, *1 2 2 3* or *1 2 3 3*, and q the number of tetrahedra of the subdivision denoted *1 2 3 4*, then, obviously,

$$A = 2p + q.$$

From the equality

$$2k + l = 2p + q$$

it follows that q and l are simultaneously even or odd. But, by proposition B, the number l is odd, and, hence, q is also odd.

In the introductory part to this section we mentioned that induction on the number of dimensions may sometimes be replaced by ordinary induction. Let us give some examples illustrating this statement.

EXAMPLE 31. Prove the proposition formulated in Example 30 A, using the method of induction on the number n of segments of the subdivision.

Solution. 1°. For $n = 1$ the assertion is obvious.

2°. Suppose that our assertion has already been proved for any division of a line segment into n smaller segments, and suppose we are given a subdivision of the line segment *1 2* into $n + 1$ smaller segments. If not all of these segments are denoted *1 2*, then there can be found a segment whose end-points are numbered by the same numbers, say *1 1*. Let us now contract this segment to a point. Then we shall get a division of the segment *1 2* into n smaller segments. By virtue of the induction assumption, in this division, and hence, also in the original subdivision as well, there can be found at least one segment denoted by the numbers *1 2*, which was required to be proved.

EXAMPLE 32. Prove the proposition formulated in Example 30 B by induction on the number n of triangles of the subdivision.

Solution. 1°. For $n = 1$ the assertion is obvious; for $n = 2$ it is easily verified.

2°. Suppose that our assertion has already been proved for any division of the triangle *1 2 3* into n or fewer triangles, and suppose we have a triangle subdivided into $n + 1$ triangles. If not all of these triangles are denoted by the number *1 2 3*, then there can be found

a triangle in which two vertices have the same numbers, say *1*. Adjacent to the side *1 1* are either two small triangles (if this side is inside the initial triangle, Fig. 77, *a*), or one triangle (if this side lies on a side of the subdivided triangle, Fig. 77, *b*). Contracting the segment *1 1* into a point, we obtain a new division of the triangle *1 2 3* into $n-1$ (in the first case, Fig. 77, *c*) or n (in the second case, Fig. 77, *d*) triangles. By virtue of the

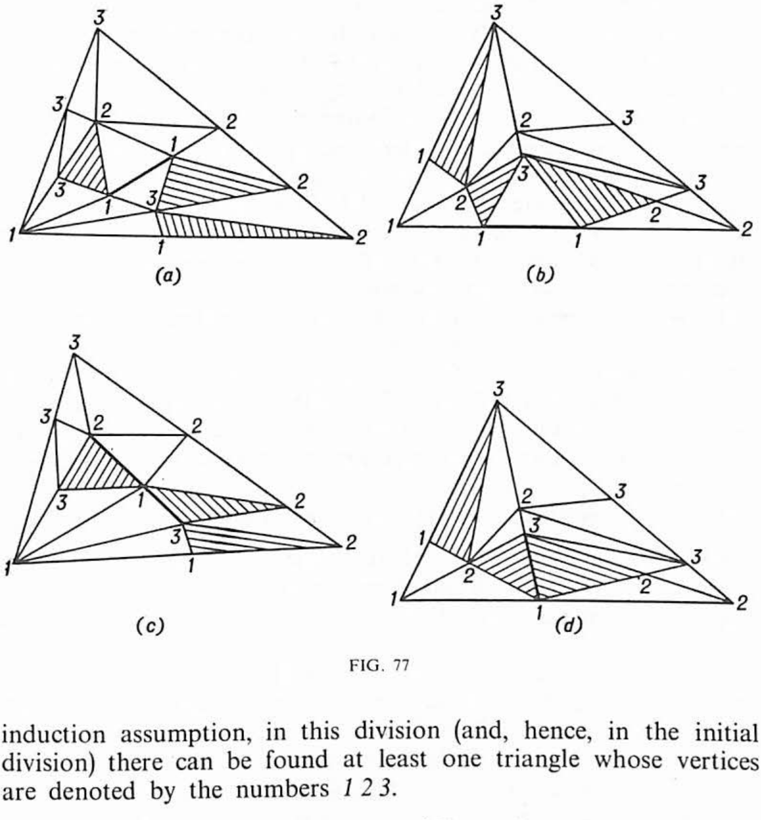


FIG. 77

induction assumption, in this division (and, hence, in the initial division) there can be found at least one triangle whose vertices are denoted by the numbers *1 2 3*.

PROBLEM 39. Prove the theorem of Example 30 C by induction on the number n of tetrahedra obtained in the subdivision.

Hint. The proof is analogous to that of the proposition in Example 32.

The proposition of Example 30 can be made still more precise. Let us introduce the notion of the "orientation" of the tetrahedron

whose vertices are numbered *1 2 3 4*, differentiating between the tetrahedra for which motion around the face *1 2 3* from vertex *1* to vertex *2* and then to vertex *3* is seen from vertex *4* as being clockwise, and the tetrahedra for which this motion is observed from vertex *4* as taking place in an anticlockwise direction. Then we have the following proposition.

PROBLEM 40. Prove that under the conditions of Example 30 the number of obtained tetrahedra numbered *1 2 3 4* and oriented in the same way as the original tetrahedron will be one more than the number of tetrahedra *1 2 3 4* oriented in the opposite way.

Consider successively the following problems.

A. Under the conditions of Example 30 A we shall distinguish between the line segments *1 2* for which the direction from vertex *1* to vertex *2* coincides with the direction from *1* to *2* of the basic segment and the line segments *1 2* for which the direction from *1* to *2* is opposite to the direction of the basic segment. Prove that the number of segments of the first type is one more than the number of segments of the second type.

B. We shall speak of the triangle *1 2 3* (see Example 30 B) as being oriented clockwise (or anticlockwise) if it is traversed from vertex *1* to vertex *2* and then to vertex *3* clockwise (or anti-clockwise). Prove that the number of triangles of the subdivision, numbered *1 2 3* and oriented in the same way as the subdivided triangle, is one more than the number of remaining small triangles numbered *1 2 3*.

C. The proposition formulated at the beginning of this example.

EXAMPLE 33. Given: n spheres in space each four of which intersect. Prove that all these spheres intersect, i. e. there exists a point belonging to all the spheres. Consider successively the following problems.

A. Given: n segments on a straight line each two of which intersect. Prove that all the segments intersect, i. e. there exists a point belonging to all of them.

Solution. 1°. For $n = 2$ the proposition is obvious.

2°. Suppose that our statement has already been proved for any n segments, and suppose $n + 1$ pairwise intersecting segments l_1, l_2, ..., l_n, l_{n+1} are given on a straight line. By the induction assumption, the n segments l_1, l_2, ..., l_n intersect. Let us denote by l their intersection (which will obviously be a point or a line segment), and prove that the line segment l_{n+1} intersects with l. Let us suppose that this is not so. Then there exists a point A separating l_{n+1} and l (Fig. 78, *a*). But each of the segments

$l_1, l_2, \ldots, l_n$ contains l and, by hypothesis, intersects with the segment l_{n+1}. This means that each of these segments contains the point A, and, hence, the point A belongs to l. The contradiction proves that l_{n+1} and l intersect; their intersection belongs to all the given segments $l_1, l_2, \ldots, l_{n+1}$.

B. Given: n spheres in a plane each three of which intersect.

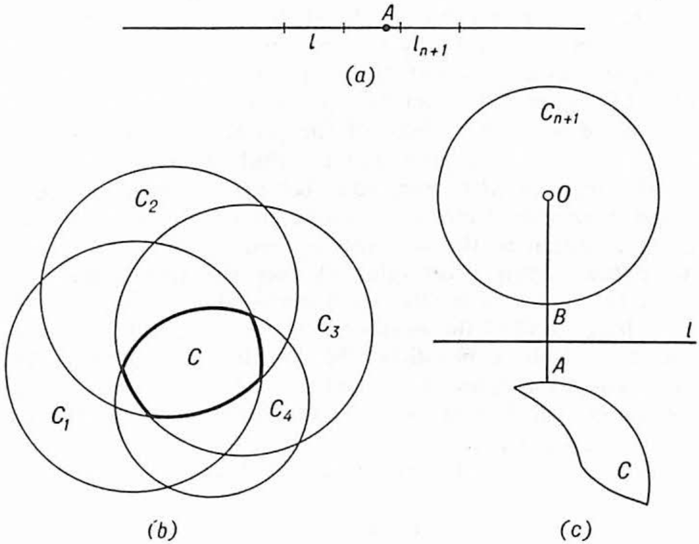


FIG. 78

Prove that there exists at least one point belonging to all the spheres.

Solution. 1°. For $n = 3$ the assertion is obvious.

2°. Suppose that our assertion has already been proved for any n circles, and suppose $n+1$ circles $C_1, C_2, \ldots, C_n, C_{n+1}$ are given in a plane. By induction assumption, the n circles $C_1, C_2, \ldots, C_n$ intersect. Let C denote their intersection (Fig. 78, *b*) (in some cases the "circular polygon" C can be a complete circle or consist only of one point). We have to prove that the figure C and the circle C_{n+1} intersect each other. Let us suppose this is not so. Then it is possible to draw a straight line l separating the figures C_{n+1} and C. A line which does this is the line l perpendicular to the line joining the centre O of the circle C_{n+1}

to the point A of the figure C, nearest to O, which passes through the midpoint of the segment AB, where B is the point of intersection of the segment OA and the circle C_{n+1} (Fig. 78, c)*.

Since each of the circles $C_1, C_2, \ldots, C_n$ contains the figure C and, by hypothesis, intersects the circle C_{n+1}, it also intersects the line l. Let a_1 denote the segment along which the circle C_1 intersects the line l, a_2 the segment along which the circle C_2 intersects this line, and so on. We shall then have n segments $a_1, a_2, \ldots, a_n$ on the line l, any two of which intersect. Consider two of them, say, a_1 and a_2. Let M be an arbitrary point of the figure C (in which case this point belongs to both circles C_1 and C_2). Furthermore, since any three of the given circles intersect, there exists a point N belonging simultaneously to C_1, C_2 and C_{n+1}. Thus the segment MN completely belongs to the circles C_1 and C_2, and, hence, the point of its intersection with the line l will be a point common to the segments a_1 and a_2.

As follows from proposition A, on the line l there exists a point belonging to all the segments $a_1, a_2, \ldots, a_n$. This point must belong to all of the circles $C_1, C_2, \ldots, C_n$ and, hence, to the figure C, which contradicts the construction of the line l. Consequently, the figures C_{n+1} and C must have at least one point in common, which will be a point common to all the circles $C_1, C_2, \ldots, C_n, C_{n+1}$.

C. The proposition formulated at the beginning of the present example.

Solution. 1°. For $n = 4$ the assertion is obvious.

2°. Suppose that our assertion has already been proved for any n spheres, and we are given $n+1$ spheres $\Phi_1, \Phi_2, \ldots, \Phi_n, \Phi_{n+1}$. Let Φ denote the intersection of the n spheres $\Phi_1, \Phi_2, \ldots, \Phi_n$ (existing by virtue of the induction assumption). Then we can show, in the same way as in Example 33 B, that if the sphere Φ_{n+1} does not intersect Φ, then there exists a plane π which separates them. The figures along which each of the spheres $\Phi_1, \Phi_2, \ldots, \Phi_n$

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* In fact, if the line l does not separate the figures C_{n+1} and C, then we can find on it a point K belonging to the figure C. In the triangle OAK the angle OAK is acute; besides, by definition of the point A. $OA \leqslant OK$. Hence, the foot L of the perpendicular dropped from O onto the straight line AK will lie between the points A and K. Since both points A and K belong to all the circles $C_1, C_2, \ldots, C_n$, the whole segment AK and, consequently, the point L on it also belong to each of the circles $C_1, C_2, \ldots, C_n$. Hence, L belongs to C. Therefore, we must have $OL \geqslant OA$. This contradiction ("a perpendicular is longer than an inclined line") proves our assertion.

intersects the plane π are circles, any three of which intersect. Consequently, in the plane π there exists a point belonging to all these circles and, hence, belonging to Φ which contradicts the definition of the plane π.

The proposition of Example 33 can also be proved by direct induction on the number of figures.

EXAMPLE 34. Prove the proposition of Example 33 B by induction on the number of circles.

Solution. Let us prove the proposition for so-called "circular polygons", i. e. for figures each of which is formed by an intersection of a finite number of circles. Our initial proposition then follows as a particular case.

1°. For $n = 3$ the proposition is obvious.

Let there be given four circular polygons C_1, C_2, C_3, C_4 each three of which intersect. Let A_1 denote a point common to the figures C_2, C_3 and C_4, A_2 a point common to the figures C_1, C_3 and C_4, and so on. Two cases are then possible:

(a) One of the points A_1, A_2, A_3, A_4, for instance, A_4 belongs to the triangle formed by the remaining three points (Fig. 79, *a*).

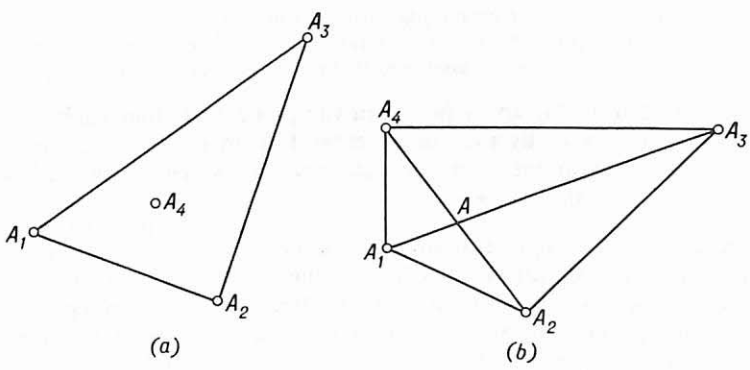

FIG. 79

Since the whole triangle $A_1A_2A_3$ belongs to C_4, the point A_4 also belongs to C_4 and so A_4 will be a point common to all four figures C_1, C_2, C_3, C_4.

(b) None of the points A_1, A_2, A_3, A_4 belongs to the triangle formed by the remaining points. In this case the point A of intersection of the diagonals of a convex quadrilateral $A_1A_2A_3A_4$ (Fig. 79, *b*) will be common to all the four figures C_1, C_2, C_3, C_4

as a common point of the triangles $A_1A_2A_3$, $A_1A_2A_4$, $A_1A_3A_4$ and $A_2A_3A_4$.

2°. Suppose our assertion is already proved for n circular polygons. Consider $n+1$ circular polygons $C_1, C_2, \ldots, C_n, C_{n+1}$. Let us denote the intersection of the figures C_n and C_{n+1} by C (it is obvious that C will also be a circular polygon), and prove that any three of the n figures $C_1, C_2, \ldots, C_{n-1}, C$ intersect. Indeed, if C is not one of these three figures, then they intersect by hypothesis. Let us now consider any three of these figures containing C, for instance, C_1 C_2, C. Since any three of the four figures C_1, C_2, C_n, C_{n+1} intersect, by virtue of 1°, these four figures have a common point which will also be common to the figures C_1, C_2 and C.

Since any three of n figures $C_1, C_2, \ldots, C_{n-1}, C$ intersect, then, by virtue of the induction assumption, all these figures have a common point which will also be common to the $n+1$ figures $C_1, C_2, \ldots, C_n, C_{n+1}$.

PROBLEM 41. Prove the proposition of Example 34 C by induction on the number n of spheres.

Hint. Prove the corresponding proposition for "spheric polygons", i.e. for solids formed by the intersection of a finite number of spheres. This proposition is proved analogously to that of Example 34.

PROBLEM 42. Given: n points in the plane such that each pair of them are at a distance of at most 1 from each other. Prove that all these points can be enclosed in a circle of radius $1/\sqrt{3}$ (*Young's Theorem*)*.

Hint. First of all show that any three of these points lie inside a circle of radius $1/\sqrt{3}$. Construct a circle of radius $1/\sqrt{3}$ with centre at each of the given points, and show that any three of the circles intersect. The point common to all these circles (existing by virtue of the result of Example 33 B) will be the centre of a circle of radius $1/\sqrt{3}$ enclosing all the given points.

PROBLEM 43. Given: n points $A_1, A_2, \ldots, A_n$ in space such that each pair of them are at a distance of at most 1 from each other. Prove that all these points can be enclosed in a sphere of radius $\sqrt{6}/4$.

Hint. The proof is analogous to the solution of Problem 42.

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* H. Young, an English mathematician of the 19th century.

A generalization of Example 33 as well as a number of applications of a more general theorem can be found in [24].

EXAMPLE 35. Let us consider a finite number of half-spaces * which fill up the whole space. Prove that from them we can select four (or fewer) half-spaces which fill the whole space.

Consider successively the following problems.

A. A one-dimensional space (a straight line) is covered by a finite number of rays. Prove that we can choose two of them which together cover the whole line.

Solution. Let A be the extreme right vertex of all the rays directed to the left, and B the extreme left vertex of all the rays directed to the right. Since, by hypothesis, the rays cover the whole line, the point B does not lie to the right of A, and two rays with their vertices at A and B completely cover the whole line.

B. The whole plane is covered with some finite number n of half-planes **. Prove that we can select two or three of them which together cover the plane.

Solution. Let us carry out the proof by induction on the number n of half-planes.

1°. For $n = 3$ the assertion is obvious.

2°. Suppose that our assertion is true for n half-planes, and let there be given $n + 1$ half-planes $F_1, F_2, \ldots, F_n, F_{n+1}$ covering the whole plane. Let us denote the boundaries of these half-planes by $l_1, l_2, \ldots, l_n, l_{n+1}$, respectively. Two cases are possible.

1st Case. The line l_{n+1} is completely contained in one of the given half-planes, say in F_n. Then the lines l_n and l_{n+1} are parallel to each other. If the half-planes F_n and F_{n+1} are situated on different sides of their boundaries (Fig. 80, *a*), then these two half-planes already cover the whole plane. Otherwise, one of them is completely contained in the other (for instance, F_{n+1} is contained in F_n; Fig. 80, *b*), and the theorem follows from the induction assumption, since in this case already n half-planes (in our case $F_1, F_2, \ldots, F_n$) cover the whole plane.

2nd Case. The line l_{n+1} is not contained in any one of the half-planes $F_1, F_2, \ldots, F_n$. Then it is completely covered by these half-planes which intersect with it in $m \leqslant n$ rays covering the whole

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* A half-space is defined as the part of space lying on one side of a plane.

** A half-plane is defined as the part of a plane lying on one side of a straight line.

line. As we saw in A, we can select two of them which also cover the whole line. The corresponding half-planes will be F_{n-1} and F_n. Let us now consider separately two possible cases of relative location of the half-planes F_{n-1}, F_n and F_{n+1}.

(a) The half-plane F_{n+1} contains the point of intersection of the lines l_{n-1} and l_n (Fig. 81, *a*). In this case the three half-planes F_{n-1}, F_n and F_{n+1} already cover the whole plane.

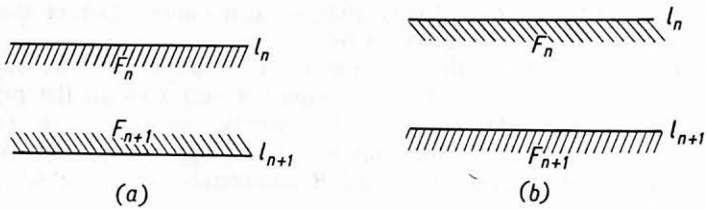

FIG. 80

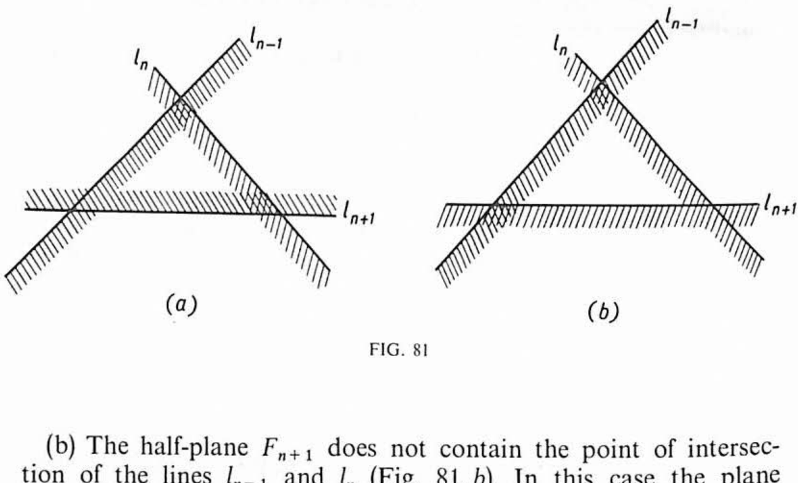

FIG. 81

(b) The half-plane F_{n+1} does not contain the point of intersection of the lines l_{n-1} and l_n (Fig. 81, *b*). In this case the plane is covered by n half-planes $F_1, F_2, \ldots, F_n$ and the theorem follows from the induction assumption.

C. The proposition formulated in the initial statement of the problem.

Solution. The proof is accomplished by induction on the number n of the given half-spaces.

1°. For $n = 4$ the assertion is obvious.

2°. Suppose that our assertion is true for n half-spaces, and let there be given $n+1$ half-spaces $V_1, V_2, \ldots, V_n, V_{n+1}$, their

boundaries being respectively denoted by $\pi_1, \pi_2, \ldots, \pi_n, \pi_{n+1}$. Two cases are then possible.

1st Case. The plane π_{n+1} is completely contained in one of the half-spaces $V_1, V_2, \ldots, V_n$ in V_n, let us say.

In this case the planes π_{n+1} and π_n are parallel. If the two half-spaces V_{n+1} and V_n lie on different sides of their boundaries, then they already fill the whole space. Otherwise, one of them is completely contained in the other, and the theorem follows from the induction assumption.

2nd Case. The plane π_{n+1} is not contained in any of the half-spaces $V_1, V_2, \ldots, V_n$. Then it is completely covered by these half-spaces, which intersect it in $m \leqslant n$ half-planes $F_1, F_2, \ldots, F_m$.

By virtue of the result obtained in B we can select two or three of these half-planes which also cover the whole plane (Figs. 80, *a* and 81, *a*). Let us consider each of the four possible cases separately.

(a) The plane π_{n+1} is covered by two half-planes (Fig. 80, *a*), say, F_1 and F_2, the corresponding planes π_1 and π_2 being parallel to each other (Fig. 82, *a*). In this case the whole space is filled by the two half-spaces V_1 and V_2.

(b) The plane π_{n+1} is covered by two half-planes F_1 and F_2, but the corresponding planes π_1 and π_2 intersect (Fig. 82, *b*). If the half-space V_{n+1} contains the line of intersection of the planes π_1 and π_2, then the three half-spaces V_1, V_2 and V_{n+1} fill the whole space. Otherwise, the half-space V_{n+1} is covered by the half-spaces V_1 and V_2, and the theorem follows from the induction assumption.

(c) The plane π_{n+1} is covered by three half-planes (Fig. 81, *a*), say, F_1, F_2 and F_3, the plane π_3 being parallel to the line of intersection of the planes π_1 and π_2 (the three planes form a "prism"; see Fig. 82, *c*). In this case the three half-spaces V_1, V_2 and V_3 fill the entire space.

(d) The plane π_{n+1} is covered with three half-planes F_1, F_2 and F_3, and the plane π_3 is not parallel to the line of intersection of π_1 and π_2 (the three planes form a "pyramid", see Fig. 82, *d*). If the half-space V_{n+1} contains the point of intersection of the planes F_1, F_2 and F_3, then the four half-spaces V_1, V_2, V_3 and V_{n+1} fill the entire space; in the contrary case, the half-space V_{n+1} is covered by the half-spaces V_1, V_2, V_3, and the theorem follows from the induction assumption.

PROBLEM 44. Prove that there cannot exist more than four rays in space, forming pairwise obtuse angles between each other.

Hint. Suppose there is given in space a finite system of rays which in pairs form obtuse angles between each other, and let us assume that this system is *maximal* in the sense that there is no ray which forms an obtuse angle with each of the given rays. Assign to each ray a half-space bounded by the plane perpendicular to the ray, and located on the same side of the plane as the ray. Since our system of rays is maximal, these half-spaces fill the entire space, and our assertion follows from the result of Example 35.

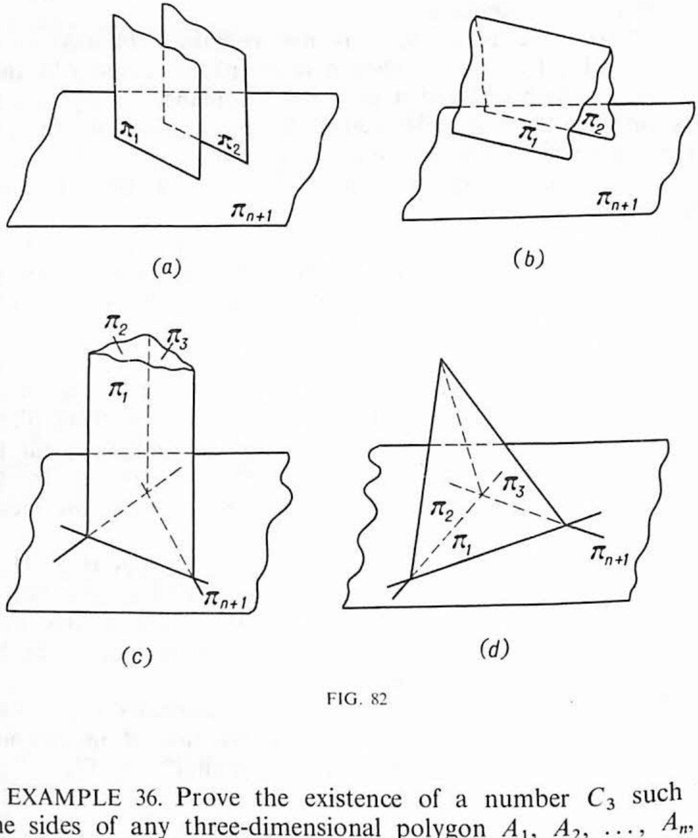

FIG. 82

EXAMPLE 36. Prove the existence of a number C_3 such that the sides of any three-dimensional polygon $A_1, A_2, \ldots, A_n$, the length of each side of which is at most 1, can be interchanged (without changing their magnitudes and directions) so that the polygon obtained as a result of this rearrangement can be enclosed in a sphere of radius C_3.

As is usual in this section, we shall first consider the corresponding "one-dimensional" and "two-dimensional" problems.

A. Given: n points $A_1, A_2, \ldots, A_n$ on a straight line, such that the length of each of the segments $A_1A_2, A_2A_3, \ldots, A_{n-1}, A_n, A_nA_1$ does not exceed 1. Prove there exists a number C_1 (not depending on the position of the points or on the number n!) such that the segments $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1$ can be rearranged on the straight line so that the polygonal line $B_1B_2 \ldots B_nB_1$ so obtained, each side of which coincides in magnitude and direction with one of the sides of the polygonal line $A_1A_2, \ldots, A_{n-1}A_nA_1$, completely lies inside a segment of length $2C_1$.

Solution. Let us take the length a_i of the link A_iA_{i+1} $(i = 1, 2, \ldots, n$; the point A_{n+1} being taken to be $A_1)$ of our polygonal line $A_1A_2 \ldots, A_nA_1$ as being positive if the point A_{i+1} is situated to the right of A_i (we consider the straight line on which all the points lie to be horizontal), and negative otherwise. Obviously, $a_1 + a_2$ is the length of the segment A_1A_3 (which may be, according to our condition, positive or negative), $a_1 + a_2 + a_3$ the length of the segment A_1A_4 and so on, $a_1 + a_2 + \ldots + a_{n-1}$ the length of the segment A_1A_n, $a_1 + a_2 + \ldots + a_{n-1} + a_n = 0$ (this being the length of the "segment A_1A_1"). Since each "link" of the polygonal line $B_1B_2 \ldots B_nB_1$ is equal to some link of the original polygonal line $A_1A_2 \ldots A_nA_1$, our proposition may be formulated in the following way:

n numbers $a_1, a_2, \ldots, a_{n-1}, a_n$ are given (positive and negative), such that each number is at most 1 in absolute value, and the sum of all numbers is zero. Prove that the numbers can be rearranged in some order $a_{i_1}, a_{i_2}, \ldots, i_{i_{n-1}}, a_{i_n}$ $(i_1, i_2, \ldots, i_{n-1}, i_n$ are the same numbers $1, 2, \ldots, n-1, n$, but rearranged in some way) *such that all the sums $a_{i_1}, a_{i_1} + a_{i_2}, a_{i_1} + a_{i_2} + a_{i_3}, \ldots, a_{i_1} + + a_{i_2} + \ldots + a_{i_{n-1}}$ do not exceed in absolute value a certain number C_1* (which is independent of the sequence $a_1, a_2, \ldots, a_n$ and even of the number n).

Let us prove that the number 1 may be taken as C_1. Let $a'_1, a'_2, \ldots, a'_p$ be the positive numbers of the sequence $a_1, a_2, \ldots, a_n$, and $a''_1, a''_2, \ldots, a''_q$ the negative numbers $(p + q = n)$. Let us take the first positive numbers $a'_1, a'_2, \ldots, a'_k$ $(k \leqslant p)$, so long as their sum does not exceed unity (for instance, one number a'_1), and then add negative numbers $a''_1, a''_2, \ldots, a''_l$ $(l \leqslant q)$ so that the sum of all the numbers taken becomes negative but is not greater than 1 in absolute value. We then proceed once again to take positive numbers and so on until all the given numbers are

the sum of any number of the first terms of the sequence obtained will not exceed 2.

Let us now arrange the numbers $\alpha'_1, \alpha'_2, \ldots, \alpha'_s$ and $\alpha''_1, \alpha''_2, \ldots, \alpha''_{n-s}$ in a common sequence so that the absolute value of the sum of any number of its first terms is not greater than 1. As is seen from the reasoning given in A, this can be done *without disturbing the sequence of the numbers* $\alpha'_1, \alpha'_2, \ldots, \alpha'_s$ *or the numbers* $\alpha''_1, \alpha''_2, \ldots, \alpha''_{n-s}$. Let the corresponding arrangement of our vectors be $\boldsymbol{a}^*_1, \boldsymbol{a}^*_2, \ldots, \boldsymbol{a}^*_n$. Then for any ν we shall have:

$$\alpha^*_1 + \alpha^*_2 + \ldots + \alpha^*_\nu \leqslant 1, \ \beta^*_1 + \beta^*_2 + \ldots + \beta^*_\nu \leqslant 2,$$

where α^*_k and β^*_k are respectively the magnitudes of the projections of $\boldsymbol{a}^*_k$ $(k = 1, 2, \ldots, n)$ on $\overline{B_1B_s}$ and on l. Since the projections of the resultant of the polygonal line formed by the vectors $\boldsymbol{a}^*_1, \boldsymbol{a}^*_2, \ldots, \boldsymbol{a}^*_\nu$ $(\nu = 1, 2, \ldots, n)$ on the straight lines $\overline{B_1B_s}$ and l are respectively equal to $\alpha^*_1 + \alpha^*_2 + \ldots + \alpha^*_\nu$ and $\beta^*_1 + \beta^*_2 + \ldots + \beta^*_\nu$, then, if the length of this resultant is equal to c_ν, we have

$$c^2_\nu = (\alpha^*_1 + \alpha^*_2 + \ldots + \alpha^*_n)^2 + (\beta^*_1 + \beta^*_2 + \ldots + \beta^*_\nu)^2,$$

i. e.

$$c^2_\nu \leqslant 5, \quad c_\nu \leqslant \sqrt{5}.$$

Thus, the distance of any vertex of the polygonal line, from a fixed point A_1, is at most $\sqrt{5}$ from which it follows that the entire obtained polygon is enclosed in a circle of radius $\sqrt{5}$.

C. The proposition formulated at the beginning of the example.

Hint. Prove that $\sqrt{21}$ may be taken as C_3. The argument here is analogous to that used in item B; $\beta_1, \beta_2, \ldots, \beta_n$ will be the projections of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ on a plane π perpendicular to the resultant $\overline{B_1B_s}$.

3. Finding Loci by Induction on the Number of Dimensions

EXAMPLE 37. Find the locus of points M in space such that the sum of the squares of the distances from them to n given points $A_1, A_2, \ldots, A_n$ is constant (and is equal to d^2).

Let us consider successively the following problems.

A. Given: n points $A_1, A_2, \ldots, A_n$ on a straight line. Find the points M on this line such that

$$MA_1^2 + MA_2^2 + \ldots + MA_n^2 = d^2,$$

where d is a given number.

Solution. Let us take our straight line to be the number axis, and let the numbers $a_1, a_2, \ldots, a_n$ correspond to the points $A_1, A_2, \ldots, A_n$, and the number x to the required point M. Then the line segments $MA_1, MA_2, \ldots, MA_n$ will be of equal length $(x - a_1), (x - a_2), \ldots, (x - a_n)$ and, consequently,

$$MA_1^2 + MA_2^2 + \ldots + MA_n^2 = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2.$$

But

$$\begin{aligned}(x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2 &= x^2 - 2a_1x + a_1^2 + x^2 - \\ &\quad - 2a_2x + a_2^2 + \ldots + x^2 - 2a_nx + a_n^2 = \\ &= nx^2 - 2(a_1 + a_2 + \ldots + a_n)x + (a_1^2 + a_2^2 + \ldots + a_n^2) = \\ &= n\left(x - \frac{a_1 + a_2 + \ldots + a_n}{n}\right)^2 + (a_1^2 + a_2^2 + \ldots + a_n^2) - \\ &\quad - \frac{(a_1 + a_2 + \ldots + a_n)^2}{n}\end{aligned}$$

or if A denotes the point corresponding to the number $\dfrac{a_1 + a_2 + \ldots + a_n}{n}$,

$$\begin{aligned}MA_1^2 + MA_2^2 + \ldots + MA_n^2 = nMA^2 + (a_1^2 + a_2^2 + \ldots + a_n^2) - \\ - \frac{(a_1 + a_2 + \ldots + a_n)^2}{n}\end{aligned} \qquad (15)$$

Thus,

$$nMA^2 = d^2 - (a_1^2 + a_2^2 + \ldots + a_n^2) + \frac{(a_1 + a_2 + \ldots + a_n)^2}{n}$$

and, hence

$$MA = \sqrt{\frac{1}{n}\left[d^2 - (a_1^2 + a_2^2 + \ldots + a_n^2) + \frac{(a_1 + a_2 + \ldots + a_n)^2}{n}\right]}.$$

This equality, generally speaking (if the radicand is positive) determines two points M satisfying the initial condition of the problem (they are located on both sides of the point A).

B. Find the locus of points M in a plane, the sum of the

the sum of any number of the first terms of the sequence obtained will not exceed 2.

Let us now arrange the numbers $\alpha'_1, \alpha'_2, \ldots, \alpha'_s$ and $\alpha''_1, \alpha''_2, \ldots, \alpha''_{n-s}$ in a common sequence so that the absolute value of the sum of any number of its first terms is not greater than 1. As is seen from the reasoning given in A, this can be done *without disturbing the sequence of the numbers* $\alpha'_1, \alpha'_2, \ldots, \alpha'_s$ *or the numbers* $\alpha''_1, \alpha''_2, \ldots, \alpha''_{n-s}$. Let the corresponding arrangement of our vectors be $\boldsymbol{a}^*_1, \boldsymbol{a}^*_2, \ldots, \boldsymbol{a}^*_n$. Then for any ν we shall have:

$$\alpha^*_1 + \alpha^*_2 + \ldots + \alpha^*_\nu \leqslant 1, \quad \beta^*_1 + \beta^*_2 + \ldots + \beta^*_\nu \leqslant 2,$$

where α^*_k and β^*_k are respectively the magnitudes of the projections of $\boldsymbol{a}^*_k$ $(k = 1, 2, \ldots, n)$ on $\overline{B_1 B_s}$ and on l. Since the projections of the resultant of the polygonal line formed by the vectors $\boldsymbol{a}^*_1, \boldsymbol{a}^*_2, \ldots, \boldsymbol{a}^*_\nu$ $(\nu = 1, 2, \ldots, n)$ on the straight lines $\overline{B_1 B_s}$ and l are respectively equal to $\alpha^*_1 + \alpha^*_2 + \ldots + \alpha^*_\nu$ and $\beta^*_1 + \beta^*_2 + \ldots + \beta^*_\nu$, then, if the length of this resultant is equal to c_ν, we have

$$c^2_\nu = (\alpha^*_1 + \alpha^*_2 + \ldots + \alpha^*_n)^2 + (\beta^*_1 + \beta^*_2 + \ldots + \beta^*_\nu)^2,$$

i. e.

$$c^2_\nu \leqslant 5, \quad c_\nu \leqslant \sqrt{5}.$$

Thus, the distance of any vertex of the polygonal line, from a fixed point A_1, is at most $\sqrt{5}$ from which it follows that the entire obtained polygon is enclosed in a circle of radius $\sqrt{5}$.

C. The proposition formulated at the beginning of the example.

Hint. Prove that $\sqrt{21}$ may be taken as C_3. The argument here is analogous to that used in item B; $\beta_1, \beta_2, \ldots, \beta_n$ will be the projections of the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ on a plane π perpendicular to the resultant $\overline{B_1 B_s}$.

3. Finding Loci by Induction on the Number of Dimensions

EXAMPLE 37. Find the locus of points M in space such that the sum of the squares of the distances from them to n given points $A_1, A_2, \ldots, A_n$ is constant (and is equal to d^2).

Let us consider successively the following problems.

A. Given: n points $A_1, A_2, \ldots, A_n$ on a straight line. Find the points M on this line such that

$$MA_1^2 + MA_2^2 + \ldots + MA_n^2 = d^2,$$

where d is a given number.

Solution. Let us take our straight line to be the number axis, and let the numbers $a_1, a_2, \ldots, a_n$ correspond to the points $A_1, A_2, \ldots, A_n$, and the number x to the required point M. Then the line segments $MA_1, MA_2, \ldots, MA_n$ will be of equal length $(x - a_1), (x - a_2), \ldots, (x - a_n)$ and, consequently,

$$MA_1^2 + MA_2^2 + \ldots + MA_n^2 = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2.$$

But

$$\begin{aligned}(x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2 &= x^2 - 2a_1x + a_1^2 + x^2 - \\ &\quad - 2a_2x + a_2^2 + \ldots + x^2 - 2a_nx + a_n^2 = \\ &= nx^2 - 2(a_1 + a_2 + \ldots + a_n)x + (a_1^2 + a_2^2 + \ldots + a_n^2) = \\ &= n\left(x - \frac{a_1 + a_2 + \ldots + a_n}{n}\right)^2 + (a_1^2 + a_2^2 + \ldots + a_n^2) - \\ &\quad - \frac{(a_1 + a_2 + \ldots + a_n)^2}{n}\end{aligned}$$

or if A denotes the point corresponding to the number $\dfrac{a_1 + a_2 + \ldots + a_n}{n}$,

$$MA_1^2 + MA_2^2 + \ldots + MA_n^2 = nMA^2 + (a_1^2 + a_2^2 + \ldots + a_n^2) - \frac{(a_1 + a_2 + \ldots + a_n)^2}{n} \qquad (15)$$

Thus,

$$nMA^2 = d^2 - (a_1^2 + a_2^2 + \ldots + a_n^2) + \frac{(a_1 + a_2 + \ldots + a_n)^2}{n}$$

and, hence

$$MA = \sqrt{\frac{1}{n}\left[d^2 - (a_1^2 + a_2^2 + \ldots + a_n^2) + \frac{(a_1 + a_2 + \ldots + a_n)^2}{n}\right]}.$$

This equality, generally speaking (if the radicand is positive) determines two points M satisfying the initial condition of the problem (they are located on both sides of the point A).

B. Find the locus of points M in a plane, the sum of the

squares of the distances to which from n given points $A_1, A_2, \ldots, A_n$ is constant (and equals d^2).

*Solution**. Let us take an arbitrary rectangular system of coordinates in the plane and denote the projections of the points $A_1, A_2, \ldots, A_n$ on the x- and y-axis by $A'_1, A'_2, \ldots, A'_n$ and $A''_1, A''_2, \ldots, A''_n$, respectively; let then M' and M'' denote the corresponding projections of the point M on the coordinate axes. In this case

$$MA_1^2 = M'A_1'^2 + M''A_1''^2,$$
$$MA_2^2 = M'A_2'^2 + M''A_2''^2,$$
$$\cdots\cdots\cdots$$
$$MA_n^2 = M'A_n'^2 + M''A_n''^2 .$$

and, hence,

$$MA_1^2 + MA_2^2 + \ldots + MA_n^2 = (M'A_1'^2 + M'A_2'^2 + \ldots + M'A_n'^2) + \\ + (M''A_1''^2 + M''A_2''^2 + \ldots + M''A_n''^2).$$

But by formula (15)

$$M'A_1'^2 + M'A_2'^2 + \ldots + M'A_n'^2 = \\ = nM'A'^2 + (a_1^2 + a_2^2 + \ldots + a_n^2) - \frac{(a_1 + a_2 + \ldots + a_n)^2}{n},$$

$$M''A_1''^2 + M''A_2''^2 + \ldots + M''A_n''^2 = \\ = nM''A''^2 + (b_1^2 + b_2^2 + \ldots + b_n^2) - \frac{(b_1 + b_2 + \ldots + b_n)^2}{n},$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are the abscissas and ordinates of the points $A_1, A_2, \ldots, A_n$ while A' and A'' are points of the x- and y-axis respectively with the coordinates $\frac{a_1 + a_2 + \ldots + a_n}{n}$ and $\frac{b_1 + b_2 + \ldots + b_n}{n}$. Thus,

$$MA_1^2 + MA_2^2 + \ldots + MA_n^2 = nMA^2 + (a_1^2 + a_2^2 + \ldots + a_n^2) + \\ + (b_1^2 + b_2^2 + \ldots + b_n^2) - \frac{(a_1 + a_2 + \ldots + a_n)^2}{n} - \\ - \frac{(b_1 + b_2 + \ldots + b_n)^2}{n} \qquad (16)$$

* Another solution of this problem was outlined above (see Problem 23).

(A is a point in the plane projected on the coordinate axes to the points A' and A''), whence

$$nMA^2 = d^2 - (a_1^2 + a_2^2 + \ldots + a_n^2) - (b_1^2 + b_2^2 + \ldots + b_n^2) + \\ + \frac{(a_1 + a_2 + \ldots + a_n)^2}{n} + \frac{(b_1 + b_2 + \ldots + b_n)^2}{n}$$

and, hence

$$MA = \sqrt{\frac{1}{n}\left[d^2 - (a_1^2 + a_2^2 + \ldots + a_n^2) - (b_1^2 + b_2^2 + \ldots + b_n^2) + \frac{(a_1 + a_2 + \ldots + a_n)^2}{n} + \frac{(b_1 + b_2 + \ldots + b_n)^2}{n}\right]},$$

i. e. the required locus represents a circle of radius

$$\sqrt{\frac{1}{n}\left[d^2 - (a_1^2 + a_2^2 + \ldots + a_n^2) - (b_1^2 + b_2^2 + \ldots + b_n^2) + \frac{(a_1 + a_2 + \ldots + a_n)^2}{n} + \frac{(b_1 + b_2 + \ldots + b_n)^2}{n}\right]},$$

if

$$d > \sqrt{(a_1^2 + a_2^2 + \ldots + a_n^2) + (b_1^2 + b_2^2 + \ldots + b_n^2) - \frac{(a_1 + a_2 + \ldots + a_n)^2}{n} - \frac{(b_1 + b_2 + \ldots + b_n)^2}{n}},$$

it consists of one point A if

$$d = \sqrt{(a_1^2 + a_2^2 + \ldots + a_n^2) + (b_1^2 + b_2^2 + \ldots + b_n^2) - \frac{(a_1 + a_2 + \ldots + a_n)^2}{n} - \frac{(b_1 + b_2 + \ldots + b_n)^2}{n}},$$

and contains no points if

$$d < \sqrt{(a_1^2 + a_2^2 + \ldots + a_n^2) + (b_1^2 + b_2^2 + \ldots + b_n^2) - \frac{(a_1 + a_2 + \ldots + a_n)^2}{n} - \frac{(b_1 + b_2 + \ldots + b_n)^2}{n}}.$$

C. The proposition stated at the beginning of the present example.

Hint. Consider the rectangular system of coordinates x, y, z in space, project all the points on the plane xOy and on the z-axis; then make use of formulas (15) and (16).

4. Definition by Induction on the Number of Dimensions

EXAMPLE 38. Definition of the medians and the centroid of a tetrahedron.

A. The *centroid of a line segment* is defined as its midpoint.

B. The *median of a triangle* is defined as a line segment joining any of its vertices to the centroid of the opposite side. As is known, the medians of a triangle intersect in a single point which is called the centroid of the triangle.

C. The *median of a tetrahedron* is defined as a line segment joining any of its vertices to the centroid of the opposite face.

Let us prove that the medians of a tetrahedron intersect in a single point.

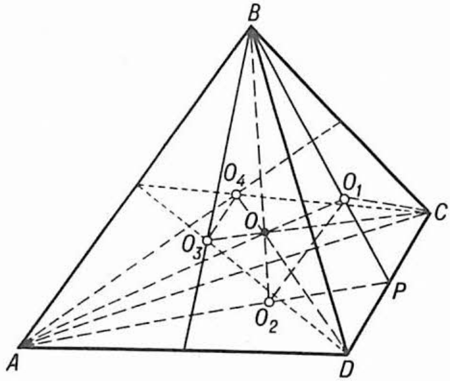

FIG. 84

Consider the tetrahedron $ABCD$ (Fig. 84) in which O_1, O_2, O_3 and O_4 are the centroids of the triangles DBC, ACD, ABD and ABC, respectively. Since the lines BO_1 and AO_2 intersect at the midpoint P of the line segment CD, the lines AO_1 and BO_2 also intersect at some point O_{12}. Analogously, AO_1 and CO_3, AO_1 and DO_4, BO_2 and CO_3, BO_2 and DO_4, CO_3 and DO_4 intersect respectively at the points O_{13}, O_{14}, O_{23}, O_{24} and O_{34}. Let us

prove that all these points coincide (point O in the figure). Suppose, for instance, that O_{12} and O_{13} do not coincide, then AO_1, BO_2 and CO_3 would lie in one plane π (i. e. in the plane $O_{12}O_{13}O_{23}$). But in this case DO_4, intersecting AO_1, BO_2 and CO_3, would also lie in the same plane, i. e. all four vertices of the tetrahedron would lie in one plane π. Since this is not the case, the points O_{12} and O_{13} must coincide. All the remaining points O_{14}, O_{23}, O_{24}, O_{34} also coincide with this point.

The point of intersection of the medians in a tetrahedron is called the *centroid of the tetrahedron.*

PROBLEM 45. Prove that the centroid of the tetrahedron divides each of its medians in the ratio $3:1$ along the median.

Hint. Make use of the fact that the centroid of the triangle divides its medians in the ratio $2:1$ (cf. Example 27).

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